

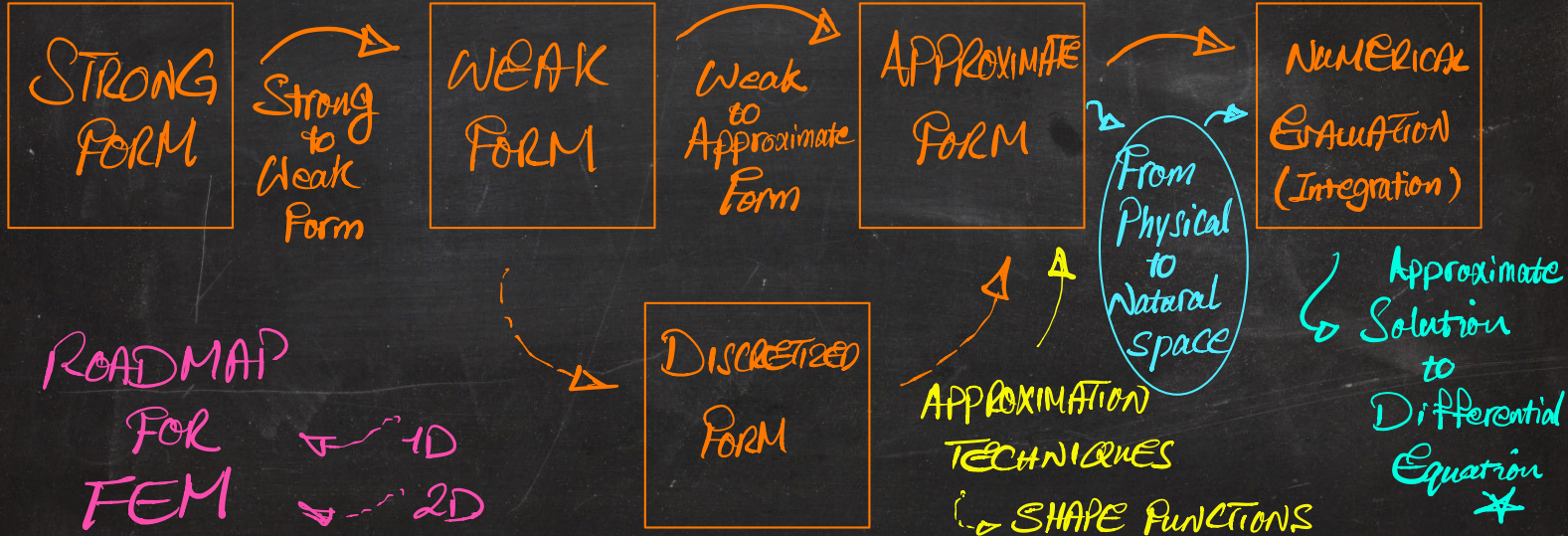
FINITE ELEMENT METHOD

FINITE ELEMENT METHOD

21

FINITE ELEMENT METHOD

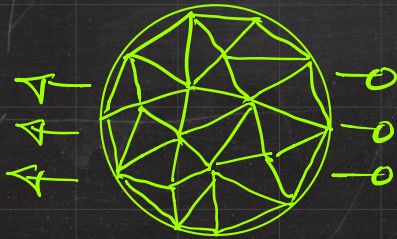
Differential
Equation ★
↓



UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)

Approximations in FEM

- Solution Approximation → inherent to numerical techniques
- Equation Approximation → diff equation using computers
- Input Approximation → space discretization is transformed to weak form + space approximation



Domain (X)

Discretization (Approximation)

→ Solution (u)

TEST (w)

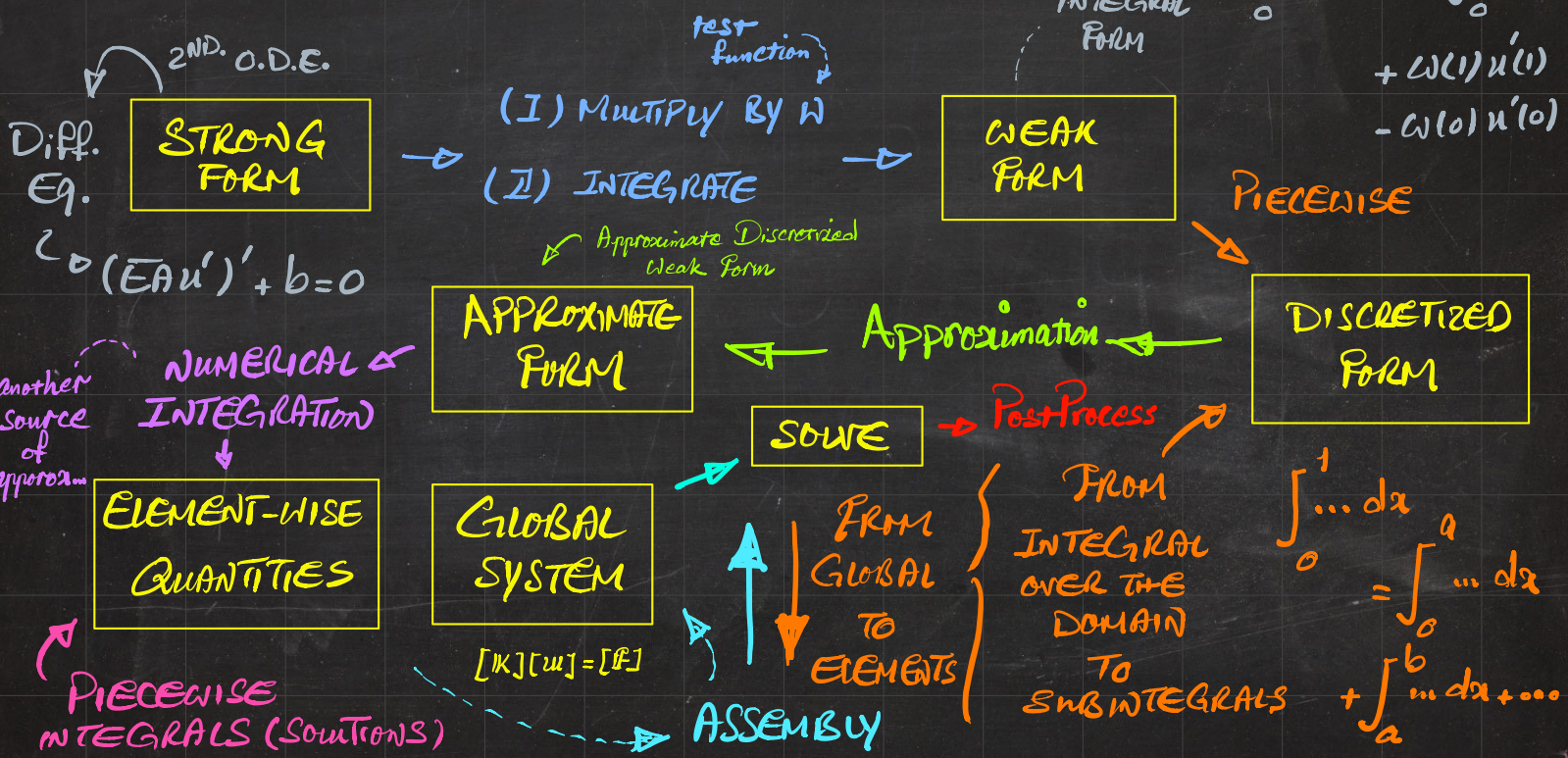
diff. Eq.

→ STRONG FORM

→ WEAK FORM

integral

FINITE ELEMENT METHOD IN A NUTSHELL



1D FEM

Overview and Wrap-up

FROM STRONG FORM TO ELEMENT STIFFNESS IN PHYSICAL SPACE now

$$EA u'' = 0 \quad \text{SUBJECT TO BCs}$$



MULTIPLY BY
TEST
FUNCTION
 w

$\left\{ \begin{array}{l} \text{DIRICHLET} \rightarrow u \text{ is PRESCRIBED} \\ \text{NEUMANN} \rightarrow u' \text{ is PRESCRIBED} \end{array} \right.$

$$EA w u'' = 0 \quad \leftarrow \text{mm} \quad w u'' = (w u')' - w' u'$$

$$EA [(w u')' - w' u'] = 0 \Rightarrow EA w' u' = EA (w u')' \quad \leftarrow \text{INTEGRATE}$$

$$\int_L EA w' u' dx = \int_L EA (w u')' dx = EA w u' \Big|_{(1)}^{(2)} = EA w^2 u'^2 - EA w^1 u'^1$$

FROM STRONG FORM TO ELEMENT STIFFNESS $\hookrightarrow$ IN PHYSICAL SPACE NOW

$EA u'' = 0$ SUBJECT TO BCs



$$EA \begin{bmatrix} \int_L N_1' N_1' dx & \int_L N_1' N_2' dx \\ \int_L N_2' N_1' dx & \int_L N_2' N_2' dx \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$\begin{bmatrix} K^{11} & K^{12} \\ K^{21} & K^{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad \hookrightarrow \quad K^{ij} = EA \int_L N_i' N_j' dx$$

$$K^{ij} = EA \int_L n^i n^j dx$$

PHYSICAL RECALL:

$$= EA \int_{-1}^1 \frac{\partial n^i}{\partial \xi} \frac{\partial n^j}{\partial \xi} \bar{J}^{-1} d\xi$$

NATURAL

$$\int_{-1}^1 g(\xi) d\xi = \sum_{gp=1}^{GPE} g(\xi) \alpha_{gp}$$

Loop over gp

$$= EA \sum_{gp=1}^{GPE} \left\{ \left[\frac{\partial n^i}{\partial \xi} \frac{\partial n^j}{\partial \xi} \bar{J}^{-1} \right]_{gp} \times \alpha_{gp} \right\}$$

END

WHAT YOU SEE IN THE CODE!

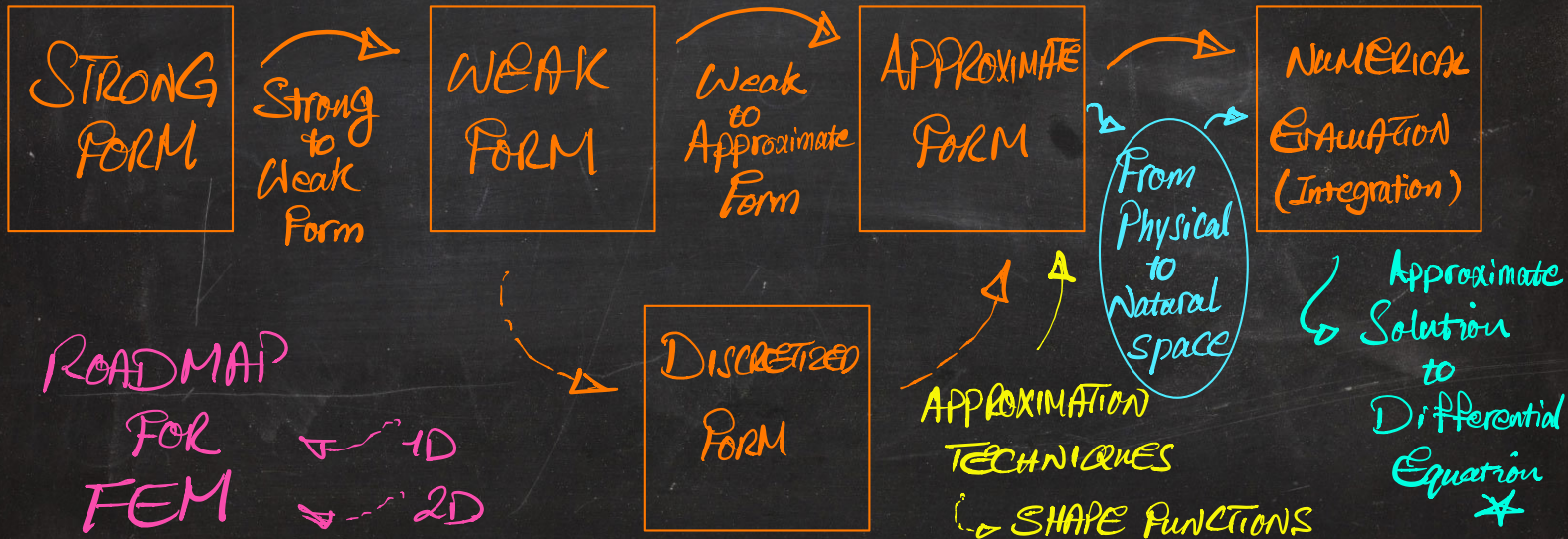
For gp=1:GPE
...
End

eq. in MATLAB

2D FEM

FINITE ELEMENT METHOD

Differential
Equation ★
↓



MATHEMATICAL PRELIMINARIES

EINSTEIN SUMMATION CONVENTION

A little definition for notation convenience

A REPEATED INDEX TWICE MEANS SUMMATION OVER THAT INDEX

also, called "dummy index"

$$\sum_{i=1}^3 u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3 \equiv u_i v_i \quad \swarrow \quad i \text{ is summation index}$$

i: free index

$$\sum_{j=1}^3 A_{ij} u_j \quad \left\{ \begin{array}{l} i=1 \Rightarrow A_{11} u_1 + A_{12} u_2 + A_{13} u_3 \\ i=2 \Rightarrow A_{21} u_1 + A_{22} u_2 + A_{23} u_3 \\ i=3 \Rightarrow A_{31} u_1 + A_{32} u_2 + A_{33} u_3 \end{array} \right. \Rightarrow \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \equiv A_{ij} u_j$$

j: summation index

$$u \cdot v = u_x v_x + u_y v_y + u_z v_z = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{i=1}^3 u_i v_i = u_i v_i$$

↳ Dot Product (u, v) $\Rightarrow$ SCALAR $\hookrightarrow u_i v_i \hookrightarrow u \cdot v$

Double Dot Product (A, B) $\Rightarrow$ SCALAR $\hookrightarrow A_{ij} B_{ij} \hookrightarrow A : B$

$u \otimes v$ Dyadic Product (u, v) $\Rightarrow$ MATRIX $\hookrightarrow u_i v_j \hookrightarrow [u \otimes v]_{ij}$

KRONECKER DELTA $\Rightarrow \delta_{ij} = \phi_i \cdot \phi_j \Rightarrow \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$$u \cdot v = u_i v_i$$

$$[A \cdot B]_{ik} = A_{ij} B_{jk}$$

$$[u \otimes v]_{ij} = u_i v_j$$

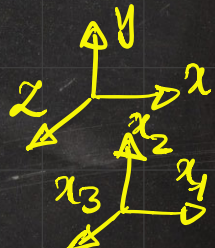
$$\delta_{ij} = \phi_i \cdot \phi_j$$

$$[A \cdot u]_i = A_{ij} u_j$$

$$A \circ B = A_{ij} B_{ij}$$

E is FOURTH-ORDER TENSOR (ARRAY) $\rightarrow 3 \times 3 \times 3 \times 3 = 81$ Components

$\swarrow$ 2nd. $\nwarrow$ 4th. $\swarrow$ 2nd.
 $B = E \circ \Phi \rightarrow [B]_{ij} = [E]_{ijkl} [\Phi]_{kl}$



INSTEAD OF $x, y, z \rightarrow 1, 2, 3 \rightarrow x_1, x_2, x_3$ $\phi_x \sim \phi_1$

DERIVATIVES $\rightarrow$ e.g. STRONG FORM $\Rightarrow u''$

$$y = f(x) \Rightarrow y' = f'(x) \quad \rightarrow \quad \{ \cdot \}' = \frac{\partial \{ \cdot \}}{\partial x}$$

$u = u(x, y, z) \Rightarrow \text{GRAD } u \quad \text{or} \quad \text{DIV } u \quad \text{or} \quad \text{CURV } u$

$$\hookrightarrow \frac{\partial u}{\partial x_i} \otimes \phi_i$$

$$\hookrightarrow \frac{\partial u}{\partial x_i} \cdot \phi_i$$

$$\hookrightarrow \frac{\partial u}{\partial x_i} \times \phi_i$$

DERIVATIVES $\rightarrow$ e.g. STRONG FORM $\rightarrow u''$

$$y = f(x) \rightarrow y' = f'(x) \quad \text{and} \quad \{ \cdot \}' = \frac{\partial \{ \cdot \}}{\partial x}$$

$$\nabla \text{GRAD } u = \frac{\partial u}{\partial x_i} \otimes \phi_i$$

$$u = u(x, y, z) \rightarrow \text{GRAD } u \quad \text{or} \quad \text{Div } u$$

$$w = f(x, y, z) \rightarrow \text{GRAD } f \quad \text{and} \quad \frac{\partial f}{\partial x_i} \phi_i$$

$$\nabla \text{Div } u = \frac{\partial u}{\partial x_i} \cdot \phi_i$$

$$A = A(x, y, z) \rightarrow \text{GRAD } A \quad \text{or} \quad \text{Div } A$$

$$\hookrightarrow \frac{\partial A}{\partial x_i} \otimes \phi_i \quad \hookrightarrow \frac{\partial A}{\partial x_i} \cdot \phi_i$$

SCALAR VALUED FUNCTION $\phi(x)$ to $\phi(x_1, x_2, x_3)$

$\uparrow$
 $\phi(x, y, z)$

$$\text{GRAD } \phi = \frac{\partial \phi}{\partial x_i} \phi_i$$

$$= \frac{\partial \phi}{\partial x_1} \phi_1 + \frac{\partial \phi}{\partial x_2} \phi_2 + \frac{\partial \phi}{\partial x_3} \phi_3$$

$$= \frac{\partial \phi}{\partial x} \phi_x + \frac{\partial \phi}{\partial y} \phi_y + \frac{\partial \phi}{\partial z} \phi_z$$

$$\text{GRAD } \phi = \begin{bmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{bmatrix}$$

VECTOR-VALUED FUNCTION $u(x) \rightsquigarrow$

$$\begin{bmatrix} u_1(x_1, x_2, x_3) \\ u_2(x_1, x_2, x_3) \\ u_3(x_1, x_2, x_3) \end{bmatrix}$$

$$\text{GRAD } u = \frac{\partial u}{\partial x_i} \otimes \phi_i$$

$$= \frac{\partial (u_j \phi_j)}{\partial x_i} \otimes \phi_i$$

$$= \frac{\partial u_j}{\partial x_i} \phi_j \otimes \phi_i$$

$$\text{GRAD } u = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

VECTOR-VALUED FUNCTION $u(x)$

$$\begin{aligned}\text{Div } u &= \frac{\partial u}{\partial x_i} \cdot \phi_i \\&= \frac{\partial (u_j \phi_j)}{\partial x_i} \cdot \phi_i \\&= \frac{\partial u_j}{\partial x_i} \underbrace{\phi_j \cdot \phi_i}_{\delta_{ij}} = \frac{\partial u_i}{\partial x_i}\end{aligned}$$

$$\text{GRAD } u = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$\text{Div } u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

MATRIX-VALUED FUNCTION $A(x)$

$$\text{Dir } A = \frac{\partial A}{\partial x_i} \cdot \phi_i$$

$$= \frac{\partial (A_{jk} \phi_j \otimes \phi_k)}{\partial x_i} \cdot \phi_i$$

$$= \frac{\partial A_{jk}}{\partial x_i} \phi_j \underbrace{\phi_k \cdot \phi_i}_{\delta_{ki}} = \frac{\partial A_{ji}}{\partial x_i} \phi_j$$

$$\Rightarrow [\text{Dir } A]_j^i = \frac{\partial A_{ji}}{\partial x_i}$$

$$\text{Dir } A = \begin{bmatrix} \frac{\partial A_{11}}{\partial x_1} + \frac{\partial A_{12}}{\partial x_2} + \frac{\partial A_{13}}{\partial x_3} \\ \frac{\partial A_{21}}{\partial x_1} + \frac{\partial A_{22}}{\partial x_2} + \frac{\partial A_{23}}{\partial x_3} \\ \frac{\partial A_{31}}{\partial x_1} + \frac{\partial A_{32}}{\partial x_2} + \frac{\partial A_{33}}{\partial x_3} \end{bmatrix}$$

SCALAR $\rightarrow 0$, VECTOR $\rightarrow 1$, MATRIX $\rightarrow 2$

$$\text{GRAD } \Phi = \begin{bmatrix} \partial \Phi / \partial x_1 \\ \partial \Phi / \partial x_2 \\ \partial \Phi / \partial x_3 \end{bmatrix}$$

$$\text{GRAD } u = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

$$\text{DIV } u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

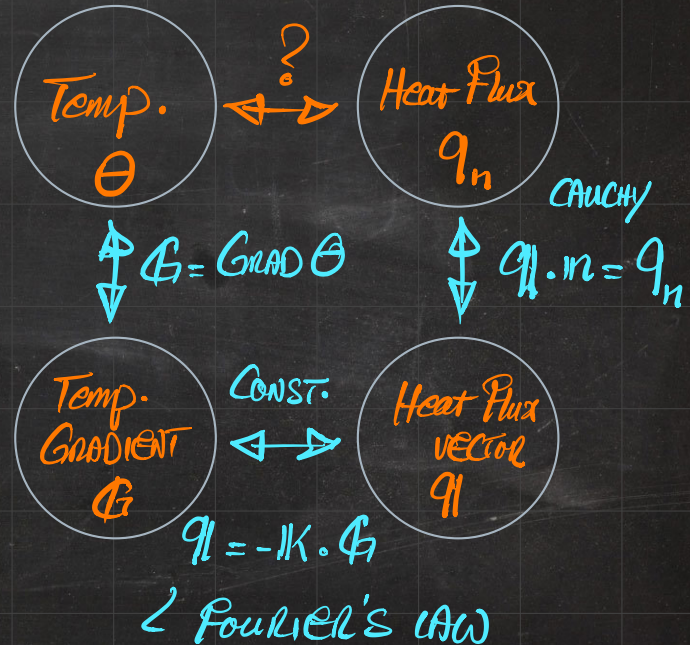
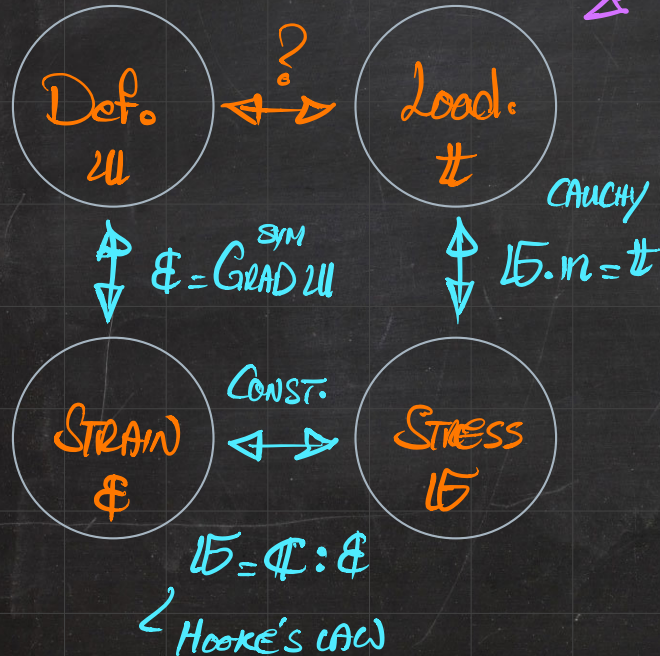
$$\text{DIV } A =$$

$$\begin{bmatrix} \frac{\partial A_{11}}{\partial x_1} + \frac{\partial A_{12}}{\partial x_2} + \frac{\partial A_{13}}{\partial x_3} \\ \frac{\partial A_{21}}{\partial x_1} + \frac{\partial A_{22}}{\partial x_2} + \frac{\partial A_{23}}{\partial x_3} \\ \frac{\partial A_{31}}{\partial x_1} + \frac{\partial A_{32}}{\partial x_2} + \frac{\partial A_{33}}{\partial x_3} \end{bmatrix}$$

GRADIENT INCREASES
THE ORDER BY 1

DIVERGENCE REDUCES
THE ORDER BY 1

Big Picture of Mechanics (Mechanical Problems & Thermal Problems)



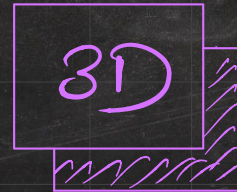
STRONG FORM (Generic Form) $\rightarrow \text{Div } \mathbf{b} + \mathbf{b} = 0$, $\text{Div } \mathbf{q} + c = 0$

$$\frac{\partial b_{jk}}{\partial x_k} + b_j = 0$$

$$\frac{\partial q_i}{\partial x_i} + c = 0$$

$$\left\{ \begin{array}{l} \frac{\partial b_{xx}}{\partial x} + \frac{\partial b_{xy}}{\partial y} + \frac{\partial b_{xz}}{\partial z} + b_x = 0 \\ \frac{\partial b_{yx}}{\partial x} + \frac{\partial b_{yy}}{\partial y} + \frac{\partial b_{yz}}{\partial z} + b_y = 0 \\ \frac{\partial b_{zx}}{\partial x} + \frac{\partial b_{zy}}{\partial y} + \frac{\partial b_{zz}}{\partial z} + b_z = 0 \end{array} \right.$$

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} + c = 0$$



STRONG FORM (Generic Form) $\rightarrow \text{Div } \mathbf{B} + \mathbf{b} = 0$, $\text{Div } \mathbf{q} + c = 0$

$$\frac{\partial B_{jk}}{\partial x_k} + b_j = 0$$

$$\frac{\partial q_i}{\partial x_i} + c = 0$$

$$\begin{cases} \frac{\partial B_{xx}}{\partial x} + \frac{\partial B_{xy}}{\partial y} + b_x = 0 \\ \frac{\partial B_{yx}}{\partial x} + \frac{\partial B_{yy}}{\partial y} + b_y = 0 \end{cases}$$

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + c = 0$$

2D $\Rightarrow$ Plane STRAIN
Plane STRESS } $\mathbf{B} = \mathbf{\Phi} : \mathbf{\Phi}$



APPROXIMATION USING 2D FINITE ELEMENTS $\rightarrow$ Shape Functions

EXAMPLE 1:

$$f(-1, -1) = 1$$

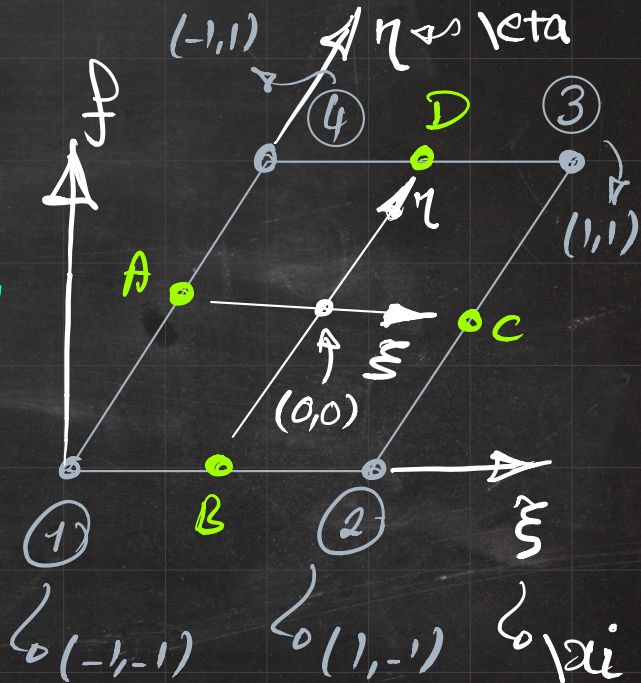
$$f(1, -1) = 2$$

$$f(1, 1) = -1$$

$$f(-1, 1) = 4$$

$$\hookrightarrow f(0, 0) = ? , f(\xi, \eta) = ?$$

$$f(\xi, \eta) = N^T f^T$$
$$+ N^2 f^2$$
$$+ N^3 f^3$$
$$+ N^4 f^4$$



APPROXIMATION USING 2D FINITE ELEMENTS $\rightarrow$ Shape Functions

EXAMPLE 1:

$$f(\xi, \eta) = N^i f^i$$

$$N^1(\xi, \eta) = \frac{1}{4} [\xi - 1][\eta - 1]$$

$$N^2(\xi, \eta) = -\frac{1}{4} [\xi + 1][\eta - 1]$$

$$N^3(\xi, \eta) = \frac{1}{4} [\xi + 1][\eta + 1]$$

$$N^4(\xi, \eta) = -\frac{1}{4} [\xi - 1][\eta + 1]$$

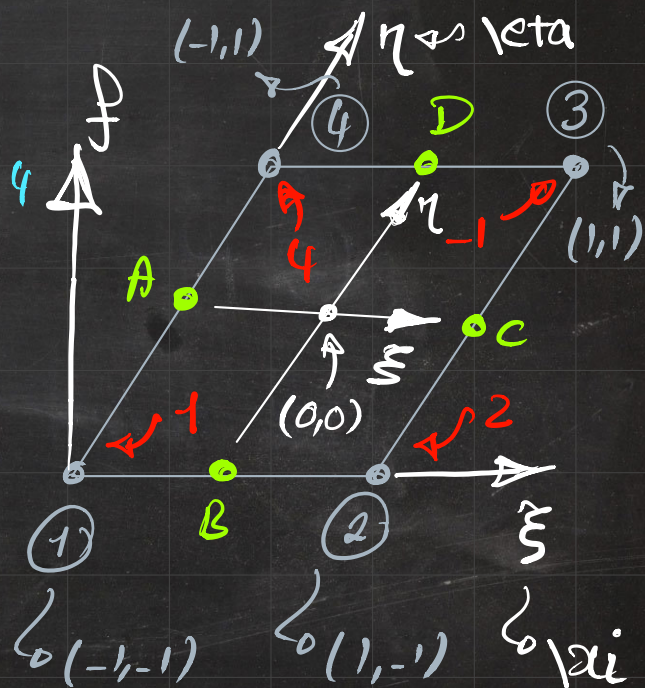
$$= \frac{1}{4} [\xi - 1][\eta - 1] \times 1$$

$$- \frac{1}{4} [\xi + 1][\eta - 1] \times 2$$

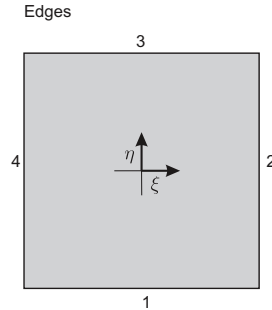
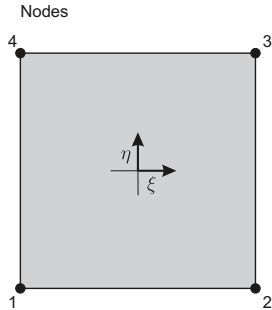
$$+ \frac{1}{4} [\xi + 1][\eta + 1] \times -1$$

$$- \frac{1}{4} [\xi - 1][\eta + 1] \times 4$$

$$= 0 = f(\xi, \eta) \checkmark$$



- two-dimensional 4-noded quadrilateral element (D2QU4N)
a.k.a. bilinear quadrilateral element
- two-dimensional 9-noded quadrilateral element (D2QU9N)
a.k.a. Lagrange biquadratic quadrilateral element
- two-dimensional 8-noded quadrilateral element (D2QU8N)
a.k.a. serendipity biquadratic quadrilateral element
- two-dimensional 3-noded triangular element (D2TR3N)
a.k.a. constant strain triangle
- two-dimensional 6-noded triangular element (D2TR6N)
a.k.a. quadratic triangle
- two-dimensional quadrature rule



Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1

$$N^1 = \frac{1}{4} (1 - \xi) (1 - \eta)$$

$$N_{,\xi}^1 = -\frac{1}{4} (1 - \eta)$$

$$N_{,\eta}^1 = -\frac{1}{4} (1 - \xi)$$

$$N^2 = \frac{1}{4} (1 + \xi) (1 - \eta)$$

$$N_{,\xi}^2 = +\frac{1}{4} (1 - \eta)$$

$$N_{,\eta}^2 = -\frac{1}{4} (1 + \xi)$$

$$N^3 = \frac{1}{4} (1 + \xi) (1 + \eta)$$

$$N_{,\xi}^3 = +\frac{1}{4} (1 + \eta)$$

$$N_{,\eta}^3 = +\frac{1}{4} (1 + \xi)$$

$$N^4 = \frac{1}{4} (1 - \xi) (1 + \eta)$$

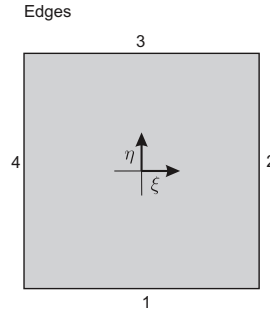
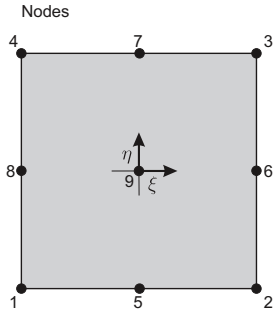
$$N_{,\xi}^4 = -\frac{1}{4} (1 + \eta)$$

$$N_{,\eta}^4 = +\frac{1}{4} (1 - \xi)$$

2D Finite Element Library

D2QU9N

Lagrange biquadratic quadrilateral element



Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1
5	0	-1
6	1	0
7	0	1
8	-1	0
9	0	0

$$N^1 = +\frac{1}{4} (1 - \xi) \xi (1 - \eta) \eta$$

$$N^2 = -\frac{1}{4} (1 + \xi) \xi (1 - \eta) \eta$$

$$N^3 = +\frac{1}{4} (1 + \xi) \xi (1 + \eta) \eta$$

$$N^4 = -\frac{1}{4} (1 - \xi) \xi (1 + \eta) \eta$$

$$N^5 = -\frac{1}{2} (1 - \xi) (1 + \xi) (1 - \eta) \eta$$

$$N^6 = +\frac{1}{2} (1 + \xi) \xi (1 - \eta) (1 + \eta)$$

$$N^7 = +\frac{1}{2} (1 - \xi) (1 + \xi) (1 + \eta) \eta$$

$$N^8 = -\frac{1}{2} (1 - \xi) \xi (1 - \eta) (1 + \eta)$$

$$N^9 = (1 - \xi) (1 + \xi) (1 - \eta) (1 + \eta)$$

$$N_{,\xi}^1 = +\frac{1}{4} (1 - 2\xi) (1 - \eta) \eta$$

$$N_{,\xi}^2 = -\frac{1}{4} (1 + 2\xi) (1 - \eta) \eta$$

$$N_{,\xi}^3 = +\frac{1}{4} (1 + 2\xi) (1 + \eta) \eta$$

$$N_{,\xi}^4 = -\frac{1}{4} (1 - 2\xi) (1 + \eta) \eta$$

$$N_{,\xi}^5 = \xi \eta (1 - \eta)$$

$$N_{,\xi}^6 = \frac{1}{2} (1 + 2\xi) (1 - \eta) (1 + \eta)$$

$$N_{,\xi}^7 = -\xi \eta (1 + \eta)$$

$$N_{,\xi}^8 = -\frac{1}{2} (1 - 2\xi) (1 - \eta) (1 + \eta)$$

$$N_{,\xi}^9 = -2\xi (1 - \eta) (1 + \eta)$$

$$N_{,\eta}^1 = +\frac{1}{4} (1 - \xi) \xi (1 - 2\eta)$$

$$N_{,\eta}^2 = -\frac{1}{4} (1 + \xi) \xi (1 - 2\eta)$$

$$N_{,\eta}^3 = +\frac{1}{4} (1 + \xi) \xi (1 + 2\eta)$$

$$N_{,\eta}^4 = -\frac{1}{4} (1 - \xi) \xi (1 + 2\eta)$$

$$N_{,\eta}^5 = \frac{1}{2} (1 - \xi) (1 + \xi) (2\eta - 1)$$

$$N_{,\eta}^6 = -(1 + \xi) \xi \eta$$

$$N_{,\eta}^7 = \frac{1}{2} (1 - \xi) (1 + \xi) (1 + 2\eta)$$

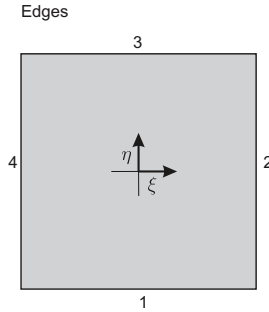
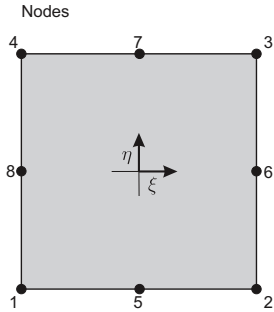
$$N_{,\eta}^8 = (1 - \xi) \xi \eta$$

$$N_{,\eta}^9 = -2(1 - \xi) (1 + \xi) \eta$$

2D Finite Element Library

D2QU8N

serendipity biquadratic quadrilateral element



Node Number	Coordinates	
	ξ	η
1	-1	-1
2	1	-1
3	1	1
4	-1	1
5	0	-1
6	1	0
7	0	1
8	-1	0

$$N^1 = -\frac{1}{4} (1 - \xi) (1 - \eta) (1 + \xi + \eta)$$

$$N^2 = -\frac{1}{4} (1 + \xi) (1 - \eta) (1 - \xi + \eta)$$

$$N^3 = -\frac{1}{4} (1 + \xi) (1 + \eta) (1 - \xi - \eta)$$

$$N^4 = -\frac{1}{4} (1 - \xi) (1 + \eta) (1 + \xi - \eta)$$

$$N^5 = \frac{1}{2} (1 - \xi) (1 + \xi) (1 - \eta)$$

$$N^6 = \frac{1}{2} (1 + \xi) (1 + \eta) (1 - \eta)$$

$$N^7 = \frac{1}{2} (1 - \xi) (1 + \xi) (1 + \eta)$$

$$N^8 = \frac{1}{2} (1 - \xi) (1 + \eta) (1 - \eta)$$

$$N_{,\xi}^1 = +\frac{1}{4} (1 - \eta) (2\xi + \eta)$$

$$N_{,\xi}^2 = +\frac{1}{4} (1 - \eta) (2\xi - \eta)$$

$$N_{,\xi}^3 = +\frac{1}{4} (1 + \eta) (2\xi + \eta)$$

$$N_{,\xi}^4 = +\frac{1}{4} (1 + \eta) (2\xi - \eta)$$

$$N_{,\xi}^5 = -\xi (1 - \eta)$$

$$N_{,\xi}^6 = +\frac{1}{2} (1 - \eta) (1 + \eta)$$

$$N_{,\xi}^7 = -\xi (1 + \eta)$$

$$N_{,\xi}^8 = -\frac{1}{2} (1 - \eta) (1 + \eta)$$

$$N_{,\eta}^1 = +\frac{1}{4} (1 - \xi) (\xi + 2\eta)$$

$$N_{,\eta}^2 = +\frac{1}{4} (1 + \xi) (-\xi + 2\eta)$$

$$N_{,\eta}^3 = +\frac{1}{4} (1 + \xi) (\xi + 2\eta)$$

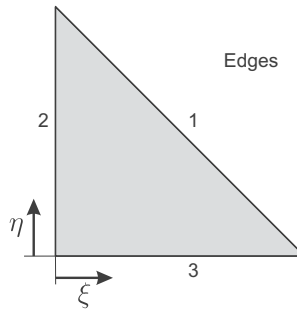
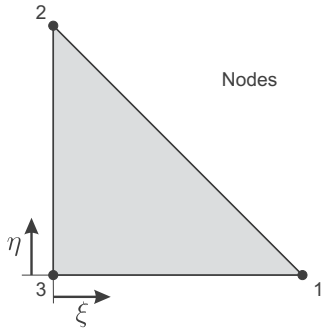
$$N_{,\eta}^4 = +\frac{1}{4} (1 - \xi) (-\xi + 2\eta)$$

$$N_{,\eta}^5 = -\frac{1}{2} (1 - \xi) (1 + \xi)$$

$$N_{,\eta}^6 = -(1 + \xi) \eta$$

$$N_{,\eta}^7 = +\frac{1}{2} (1 - \xi) (1 + \xi)$$

$$N_{,\eta}^8 = -(1 - \xi) \eta$$



Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0

$$N^1 = \xi$$

$$N^2 = \eta$$

$$N^3 = (1 - \xi - \eta)$$

$$N_{,\xi}^1 = 1$$

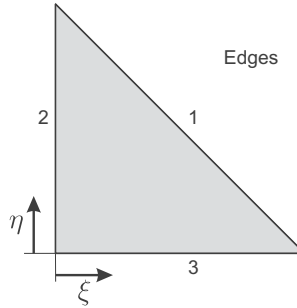
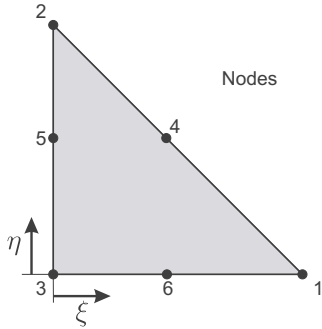
$$N_{,\eta}^1 = 0$$

$$N_{,\xi}^2 = 0$$

$$N_{,\eta}^2 = 1$$

$$N_{,\xi}^3 = -1$$

$$N_{,\eta}^3(\xi, \eta) = -1$$



Node Number	Coordinates	
	ξ	η
1	1	0
2	0	1
3	0	0
4	1/2	1/2
5	0	1/2
6	1/2	0

$$N^1 = \xi(2\xi - 1)$$

$$N_{,\xi}^1 = -1 + 4\xi$$

$$N_{,\eta}^1 = 0$$

$$N^2 = \eta(2\eta - 1)$$

$$N_{,\xi}^2 = 0$$

$$N_{,\eta}^2 = -1 + 4\eta$$

$$N^3 = (1 - \xi - \eta)(1 - 2\xi - 2\eta)$$

$$N_{,\xi}^3 = -3 + 4\xi + 4\eta$$

$$N_{,\eta}^3 = -3 + 4\xi + 4\eta$$

$$N^4 = 4\xi\eta$$

$$N_{,\xi}^4 = 4\eta$$

$$N_{,\eta}^4 = 4\xi$$

$$N^5 = 4\eta(1 - \xi - \eta)$$

$$N_{,\xi}^5 = -4\eta$$

$$N_{,\eta}^5 = -4(-1 + 2\eta + \xi)$$

$$N^6 = 4\xi(1 - \xi - \eta)$$

$$N_{,\xi}^6 = -4(-1 + \eta + 2\xi)$$

$$N_{,\eta}^6 = -4\xi$$

two-dimensional quadrature rule i

Triangular Elements Gauss Point Rule

$$\int_0^1 \int_0^{1-\eta} \{\bullet\} d\xi d\eta \approx \frac{1}{2} \sum_{i=1}^{N_{\text{Gauss Points}}} \alpha_i \{\bullet\}|_{\text{Gauss Point } i}$$

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	1/3	1/3	1

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	1/6	1/6	1/3
2	4/6	1/6	1/3
3	1/6	4/6	1/3

two-dimensional quadrature rule ii

Quadrilateral Elements Gauss Point Rule

$$\int_{-1}^1 \int_{-1}^1 \{\bullet\} d\xi d\eta \approx \sum_{i=1}^{N_{\text{Gauss Points}}} \alpha_i \{\bullet\}|_{\text{Gauss Point}^i}$$

Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	0	0	2×2

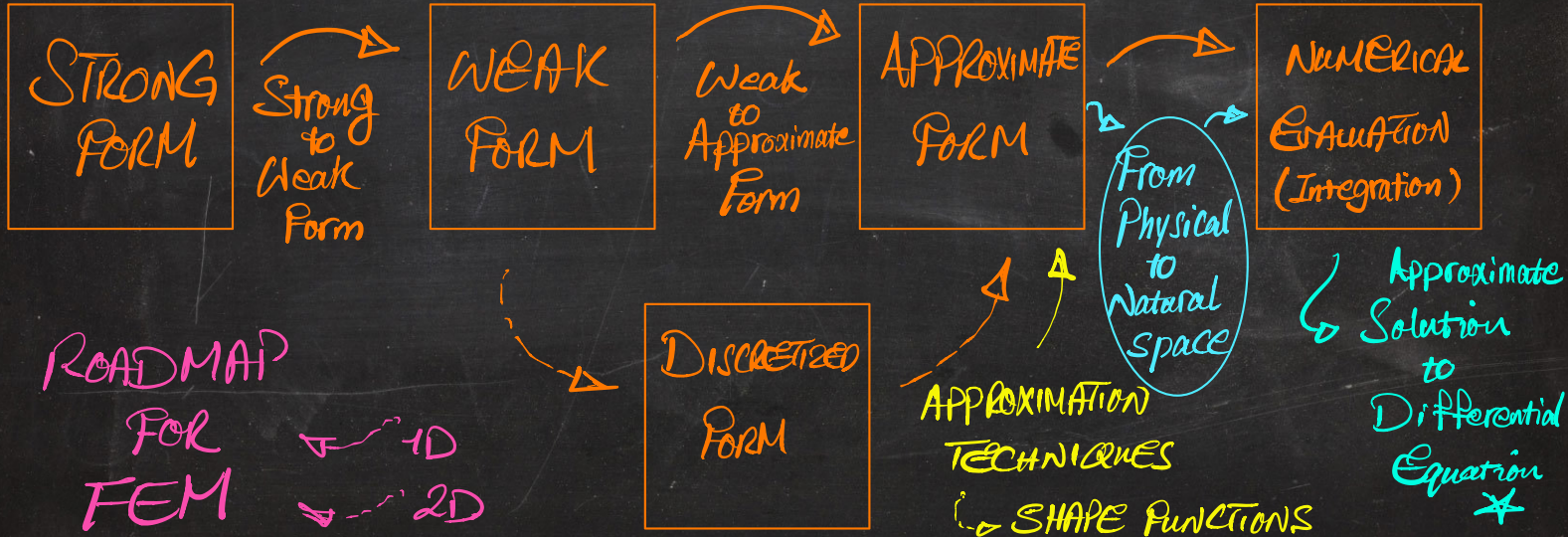
Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	$-1/\sqrt{3}$	$-1/\sqrt{3}$	1×1
2	$+1/\sqrt{3}$	$-1/\sqrt{3}$	1×1
3	$+1/\sqrt{3}$	$+1/\sqrt{3}$	1×1
4	$-1/\sqrt{3}$	$+1/\sqrt{3}$	1×1

two-dimensional quadrature rule iii

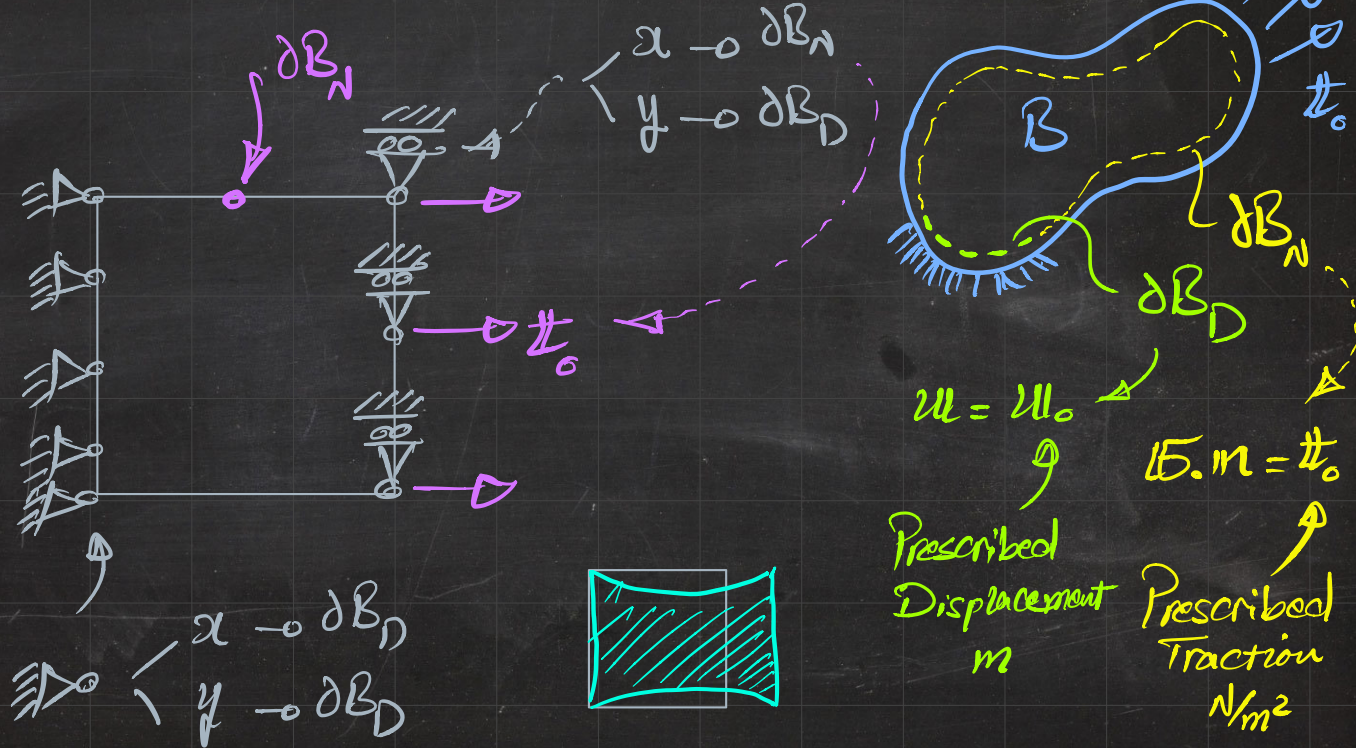
Gauss Point Number	Coordinates		Weight Factor
	ξ	η	α
1	$-\sqrt{3/5}$	$-\sqrt{3/5}$	$5/9 \times 5/9$
2	$+\sqrt{3/5}$	$-\sqrt{3/5}$	$5/9 \times 5/9$
3	$\sqrt{3/5}$	$\sqrt{3/5}$	$5/9 \times 5/9$
4	$-\sqrt{3/5}$	$\sqrt{3/5}$	$5/9 \times 5/9$
5	0	$-\sqrt{3/5}$	$5/9 \times 8/9$
6	$+\sqrt{3/5}$	0	$5/9 \times 8/9$
7	0	$\sqrt{3/5}$	$5/9 \times 8/9$
8	$-\sqrt{3/5}$	0	$5/9 \times 8/9$
9	0	0	$8/9 \times 8/9$

FINITE ELEMENT METHOD

Differential
Equation ★
↓



FROM STRONG FORM TO WEAK FORM

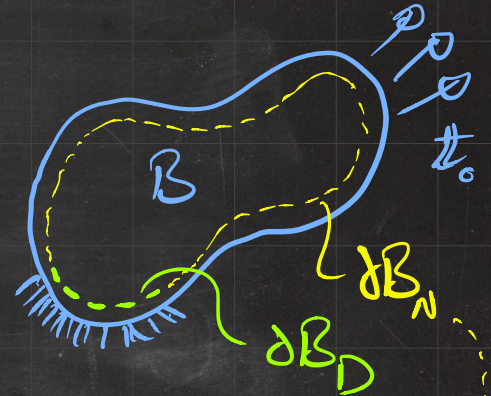


FROM STRONG FORM TO WEAK FORM

$\text{Div } \mathbf{E} = \phi$ in B subject to BCs

① STRONG FORM IN THE ABSENCE OF BODY FORCES

$u = u_0$ at ∂B_D
 $\mathbf{E} \cdot \mathbf{n} = t_0$ at ∂B_N



$w \cdot \text{Div } \mathbf{E} = 0$ ^{dot} $\nwarrow$ SCALAR

① TEST FUNCTION

$\hookrightarrow \forall w$

$w|_{\partial B_D} = 0$

$u = u_0$ $\nwarrow$
 Prescribed Displacement u
 $\mathbf{E} \cdot \mathbf{n} = t_0$ $\nwarrow$
 Prescribed Traction N/m^2

FROM STRONG FORM TO WEAK FORM

$$u = u_0 \quad \text{at } \partial B_D$$

$$B \cdot n = t_0 \quad \text{at } \partial B_N$$

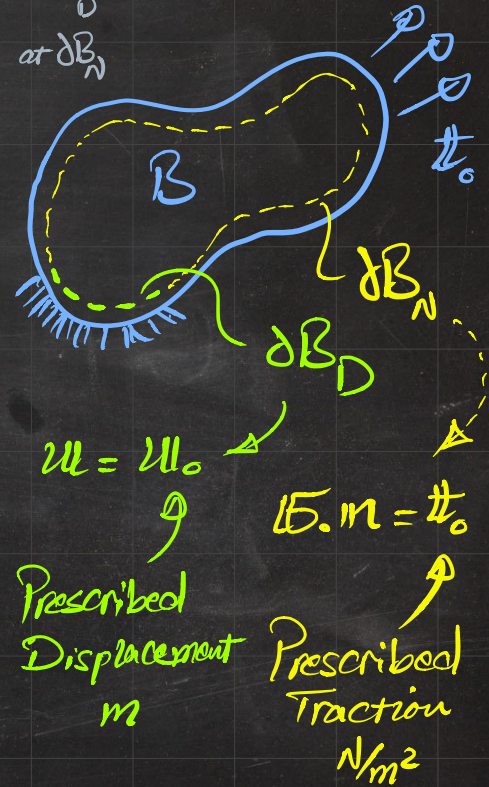
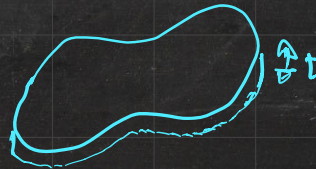
$\text{Div } B = \phi$ in B subject to BCs

w. $\text{Div } B = 0$

w. $\text{Div } B = \text{Div}(w \cdot B) - [\text{GRAD } w] : B$

$$\int_B [\text{GRAD } w] : B \, dV = \int_B \text{Div}(w \cdot B) \, dV$$

$\int_{\partial B} w \cdot t \, dA$



FROM STRONG FORM TO WEAK FORM

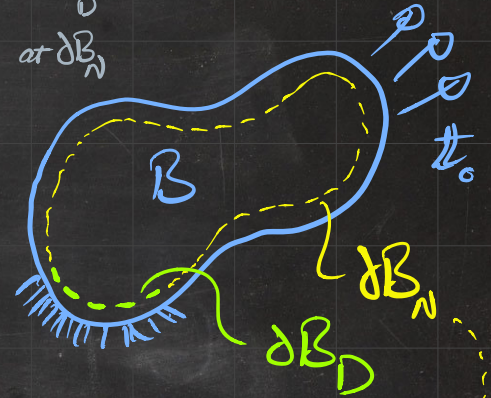
$$u = u_0 \quad \text{at } \partial B_D$$

$$B \cdot n = t_0 \quad \text{at } \partial B_N$$

$\text{Div } B = \phi$ in B subject to BCs

$$w \cdot \text{Div } B = 0$$

$$\int_B [\text{GRAD } w] : B \, dA = \int_B \text{Div } (w \cdot B) \, dA$$



$$u = u_0$$

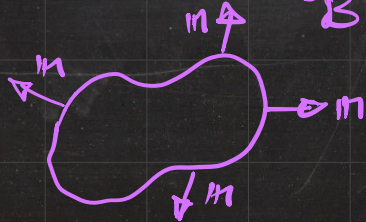
$$B \cdot n = t_0$$

Prescribed Displacement
 m

Prescribed Traction
 N/m^2

Divergence Theorem

$$\int_B \text{Div } \{ \cdot \} \, dA = \int_{\partial B} \{ \cdot \} \cdot n \, dL$$



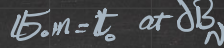
unit outward normal vector to boundary

$$u = u_0 \quad \text{at } \partial B_0$$
$$15.m = t_0 \text{ at } \delta B_N$$
$$15.m = t_0 \text{ at } \delta B_N$$

$$15.m = t_0 \text{ at } \delta B_N$$

$$15.m = t_0 \text{ at } \delta B_N$$

$$15.m = t_0 \text{ at } \delta B_N$$



FROM STRONG FORM TO WEAK FORM

$$u = u_0 \text{ at } \partial B_D$$

$$B \cdot n = t_0 \text{ at } \partial B_N$$

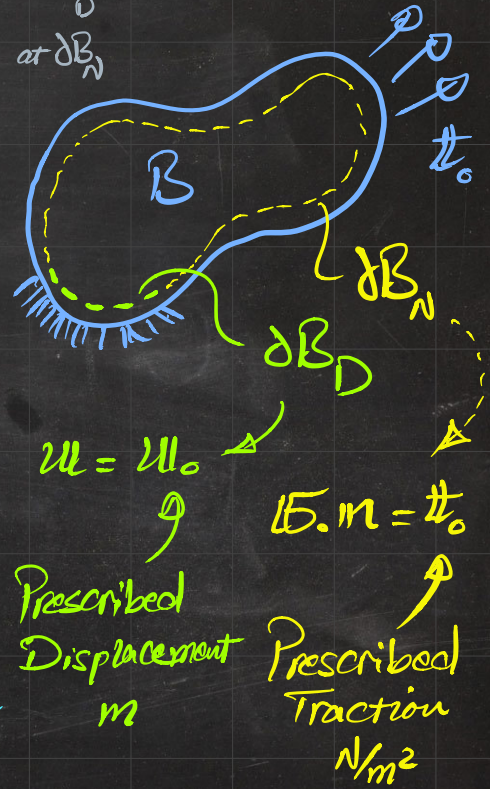
$\text{Div } B = \phi$ in B subject to BCs

w. $\text{Div } B = 0 \quad \forall w, w|_{\partial B_D} = 0$

$$\int_B [\text{GRAD } w] : B \, dA = \int_B \text{Div } (w \cdot B) \, dA$$

$$\int_B [\text{GRAD } w] : B \, dA = \int_{\partial B} w \cdot B \cdot n \, dL$$

$$= \underbrace{\int_{\partial B_D} w \cdot B \cdot n \, dL}_{=0} + \underbrace{\int_{\partial B_N} w \cdot B \cdot n \, dL}_{t_0}$$



FROM STRONG FORM TO WEAK FORM

$$u = u_0 \quad \text{at } \partial B_D$$

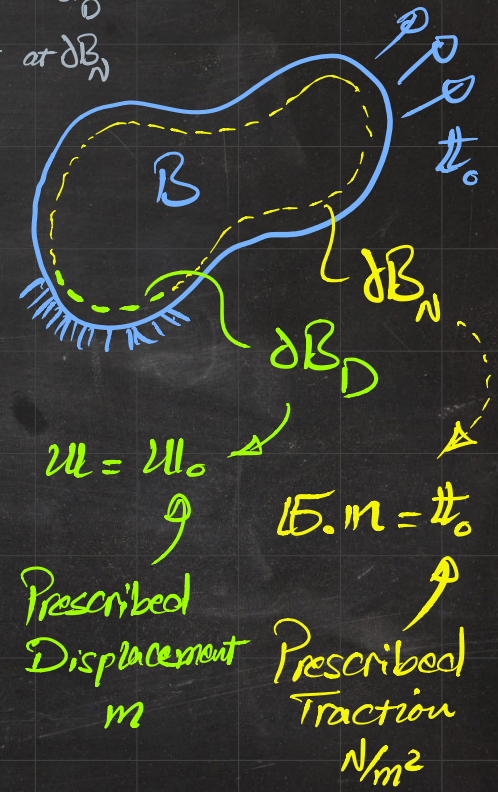
$$B \cdot n = t_0 \quad \text{at } \partial B_N$$

$\text{Div } B = \phi$ in B subject to BCs

w. $\text{Div } B = 0 \quad \forall w, w|_{\partial B_D} = 0$

$$\int_B [\text{GRAD } w] : B \, dA = \int_B \text{Div } (w \cdot B) \, dA$$

$$\int_B [\text{GRAD } w] : B \, dA = \int_{\partial B_N} w \cdot t_0 \, dL$$



FROM WEAK FORM TO APPROXIMATE FORM

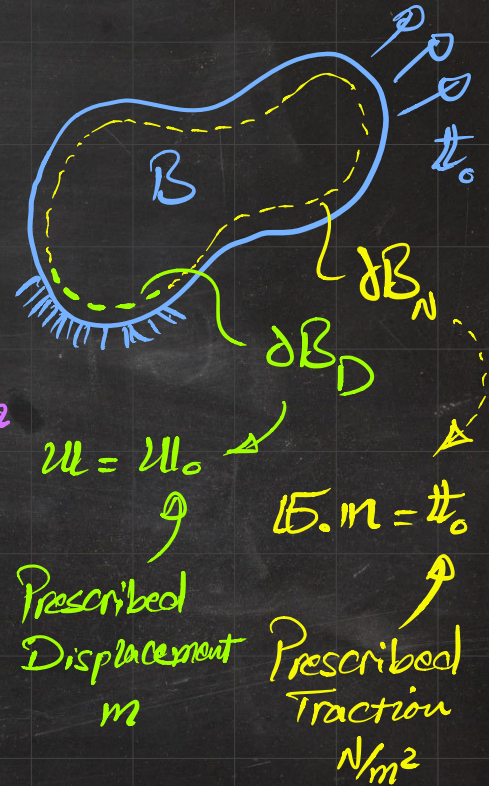
$$\int_B [\text{GRAD } w] : \mathbb{E} \, dA = \int_{\partial B_N} w \cdot \mathbb{t}_0 \, dL$$

$w^j \otimes \text{GRAD } N^j$
 $\mathbb{E} = \mathbb{E} : \Phi$
 MAJOR / MINOR — SYMMETRY —
 $\Phi = \text{GRAD } u$

$$\mathbb{E} = \mathbb{E} : [\text{GRAD } u]$$

$$w = N^j \tilde{w}^j = N^1 \tilde{w}^1 + N^2 \tilde{w}^2 + \dots$$

$$\text{GRAD } w = w^j \otimes \text{GRAD } N^j$$



FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD } w] : \mathbb{E} \, dA = \int_{\partial B_N} w \cdot \mathbf{t}_0 \, dL$$

$$w^j \otimes \text{GRAD } N^j \quad \mathbb{E} = \mathbb{E} : \mathbb{E}$$

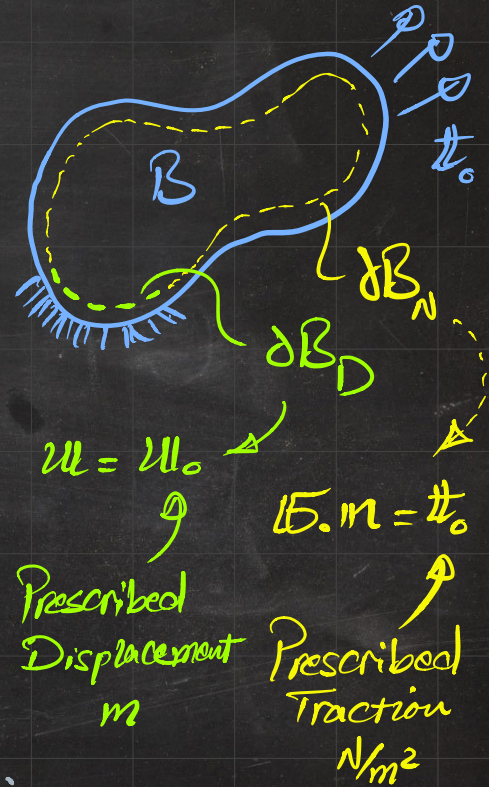
$$\mathbb{E} = \mathbb{E} : [\text{GRAD } u]$$

$$\text{GRAD } u = u^i \otimes \text{GRAD } N^i$$

$$w = N^j \tilde{w}^j$$

$$\text{GRAD } w = w^j \otimes \text{GRAD } N^j$$

$$u = N^i u^i = N^1 u^1 + N^2 u^2 + \dots$$



FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD } w] : \mathbb{E} \, dA = \int_{\partial B_N} w \cdot t_0 \, dL$$

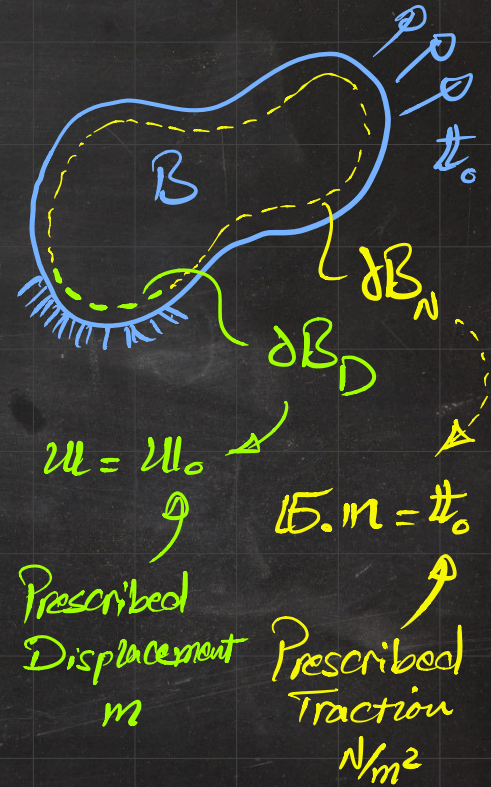
$$w^j \otimes \text{GRAD } N^j \quad \mathbb{E} = \mathbb{E} : \mathbb{E}$$

$$w = N^j \tilde{w}^j$$

$$\mathbb{E} = \mathbb{E} : [\text{GRAD } u]$$

$$\text{GRAD } w = w^j \otimes \text{GRAD } N^j$$

$$\mathbb{E} = \mathbb{E} : [u^i \otimes \text{GRAD } N^j]$$



FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD } w] : E \, dA = \int_{\partial B_N} w \cdot t_0 \, dL$$

$$w^j \otimes \text{GRAD } N^j$$

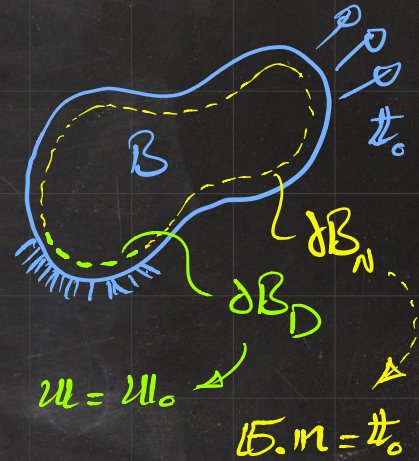
$$E = E : \mathbb{F}$$

$$w = N^j w^j$$

$$E = E : [u^i \otimes \text{GRAD } N^i]$$

$$\text{GRAD } w = w^j \otimes \text{GRAD } N^j$$

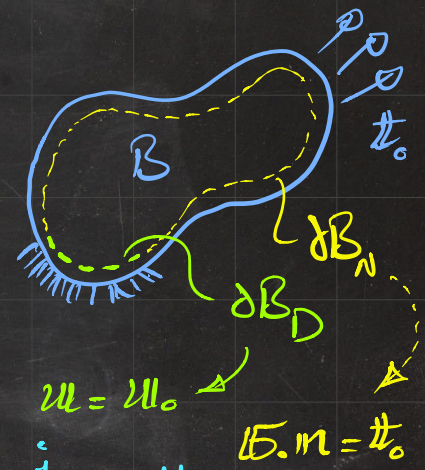
$$\int_B [w^j \otimes N^j_{,x}] : E : [u^i \otimes N^i_{,x}] \, dA = \int_{\partial B_N} N^j w^j \cdot t_0 \, dL$$



$$\text{GRAD } \{ \cdot \} = \{ \cdot \}_{,x}$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD } w] : B \, dA = \int_{\partial B_N} w \cdot t_0 \, dL$$



$$\int_B [\underbrace{w^j \otimes n_{, \alpha}^j}_{E_{abcd} \phi_a \otimes \phi_b \otimes \phi_c \otimes \phi_d} : \underbrace{\epsilon}_{\epsilon_{ijkl}} : \underbrace{u^i \otimes n_{, \alpha}^j}_{\epsilon_{ijkl}}] \, dA = \int_{\partial B_N} n^j w^j \cdot t_0 \, dL$$

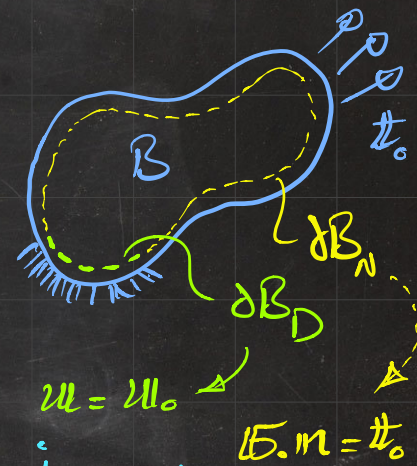
$$A : \epsilon : B$$

$\begin{matrix} \nearrow \\ 2^{nd}.O. \end{matrix}$
 $\begin{matrix} \nearrow \\ 4^{th}.O. \end{matrix}$
 $\begin{matrix} \nearrow \\ 2^{nd}.O. \end{matrix}$

$$\underbrace{A}_{ab} \underbrace{\epsilon}_{abcd} \underbrace{B}_{cd}$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD } w] : B \, dA = \int_{\partial B_N} w \cdot t_0 \, dL$$

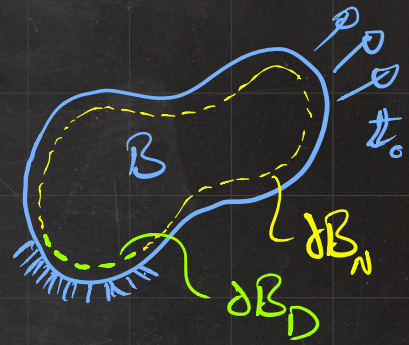


$$\int_B [w^j \otimes N_{, \alpha}^j] : \overset{E_{abcd} \phi_a \otimes \phi_b \otimes \phi_c \otimes \phi_d}{E} : [u^i \otimes N_{, \alpha}^i] \, dA = \int_{\partial B_N} N^j w^j \cdot t_0 \, dL$$

$$\int_B [w_a^j] [N_{, \alpha}^j]_b E_{abcd} [u_c^i] [N_{, \alpha}^i]_d \, dA = \int_{\partial B_N} N^j [w_a^j]_9 [t_0]_9 \, dL$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD} w] : B \, dA = \int_{\partial B_N} w \cdot t_0 \, dL$$

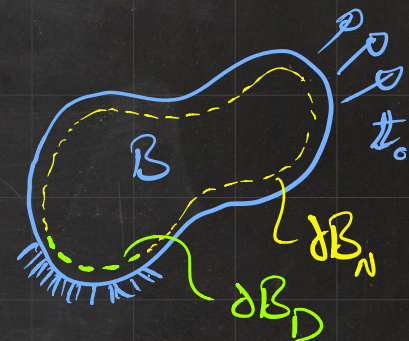


$$\int_B [w^j]_a [N^j_{,x}]_b E_{abcd} [u^i]_c [N^i_{,x}]_d \, dA = \int_{\partial B_N} N^j [w^j]_q [t_0]_q \, dL$$

$$[w^j]_a \int_B [N^j_{,x}]_b E_{abcd} [N^i_{,x}]_d \, dA [u^i]_c = [w^j]_q \int_{\partial B_N} N^j [t_0]_q \, dL$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD } w] : B \, dA = \int_{\partial B_N} w \cdot t_0 \, dL$$



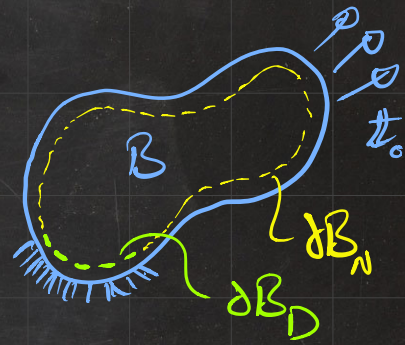
$$[w^j]_a \left[\int_B [N^j]_b \underbrace{E}_{\text{dummy}} \underbrace{[N^i]_d}_{\text{dummy}} dA \right] [u^i]_c = [w^j]_a \int_{\partial B_N} w^j [t]_q dL$$

$$[K]_{ac}^{ji} \leftarrow i, j \in \{1, 2, \dots\}$$

$$[K]_{ac}^{ji} \leftarrow a, c \in \{1, 2\}$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD} w] : B \, dA = \int_{\partial B_N} w \cdot t_0 \, dL$$



$$[w^j]_a \left[\int_B [N^j]_b E_{abcd} [N^i]_d \, dA \right] [u^i]_c = [w^j]_q \int_{\partial B_N} n^j [t_0]_q \, dL$$

$$[w^j]_a [K^{ji}]_{ac} [u^i]_c = [w^j]_q \int_{\partial B_N} n^j [t_0]_q \, dL$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD } w] : B \, dA = \int_{\partial B_N} w \cdot t_0 \, dL$$

$$[K]_{ac}^{ji} = \int_B [N]_{12a}^j{}_b E_{abcd} [N]_{12a}^i{}_d \, dA$$

$$[w^j]_a [K]_{ac}^{ji} [u^i]_c = [w^j]_a \int_{\partial B_N} N^j [t_a]_q \, dL$$

$i, j \in \{1, \dots, NPE\}$

$[w^j]$ at node j
 a in direction a
 $\hookrightarrow x, y$

$[w^1]_1 = 1$ AND ALL THE REST ARE ZERO

$$w^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad w^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad w^3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \dots$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD} w]_b : B \, dA = \int_{\partial B_N} w \cdot t_0 \, dL \quad [K]_{ac}^{ji} = \int_B [N]_{ac}^j E_{abcd} [N]_{cd}^i \, dA$$

$$[w^j]_a [K]_{ac}^{ji} [u^i]_c = [w^j]_a \int_{\partial B_N} N^j [t_a]_q \, dL$$

$$[K]_{1c}^{1i} [u^i]_c = \int_{\partial B_N} N^1 [t_a]_1 \, dL \quad \begin{matrix} F_a^1 \\ [F^1]_1 \end{matrix} \quad \begin{matrix} w^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ w^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ w^3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \dots \end{matrix}$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD } w]_b^T B \, dA = \int_{\partial B_N} w \cdot t_0 \, dL \quad [K]_{ac}^{ji} = \int_B [N]_{ac}^{ji} E_{abcd} [N]_{cd}^{ji} \, dA$$

$$[w^j]_a [K]_{ac}^{ji} [u^i]_c = [w^j]_q \int_{\partial B_N} N^j [t_0]_q \, dL$$

$$[K]_{1c}^{1i} [u^i]_c = [F]_1^1 \quad \rightsquigarrow \quad \sum_i \sum_c \dots$$

$$w^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$w^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[K]_{11}^{11} [u^1]_1 + [K]_{12}^{11} [u^1]_2 + [K]_{11}^{12} [u^2]_1 + \dots$$

$$w^3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \dots$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD } w] : B \, dA = \int_{\partial B_N} w \cdot t_0 \, dL \quad [K]_{ac}^{ji} = \int_B [N]_{ac}^j E_{abcd} [N]_{cd}^i \, dA$$

$$[w^j]_a [K]_{ac}^{ji} [u^i]_c = [w^j]_a \int_{\partial B_N} N^j [t_0]_a \, dL$$

$$[K]_{1c}^{1i} [u^i]_c = [F^1]_1 \quad \sum_i \sum_c \dots$$

$$[\dots] \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} = [F^1]_1$$

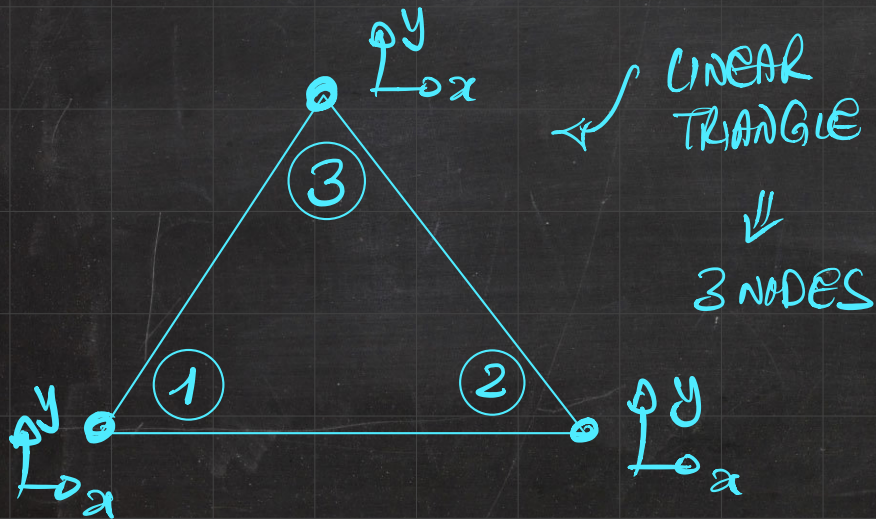
$$w^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$w^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$w^3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \dots$$

$$\begin{bmatrix} K_{11}^{11} & K_{12}^{11} & K_{11}^{12} & K_{12}^{12} & K_{11}^{13} & K_{12}^{13} \\ K_{12}^{11} & K_{22}^{11} & K_{12}^{12} & K_{22}^{12} & K_{12}^{13} & K_{22}^{13} \\ K_{11}^{12} & K_{12}^{12} & K_{11}^{21} & K_{12}^{21} & K_{11}^{22} & K_{12}^{22} \\ K_{12}^{12} & K_{22}^{12} & K_{12}^{21} & K_{22}^{21} & K_{12}^{22} & K_{22}^{22} \\ K_{11}^{13} & K_{12}^{13} & K_{11}^{22} & K_{12}^{22} & K_{11}^{31} & K_{12}^{31} \\ K_{12}^{13} & K_{22}^{13} & K_{12}^{22} & K_{22}^{22} & K_{12}^{31} & K_{22}^{31} \end{bmatrix}$$

$$\begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \\ u_1^3 \\ u_2^3 \end{bmatrix} = F_1^1$$

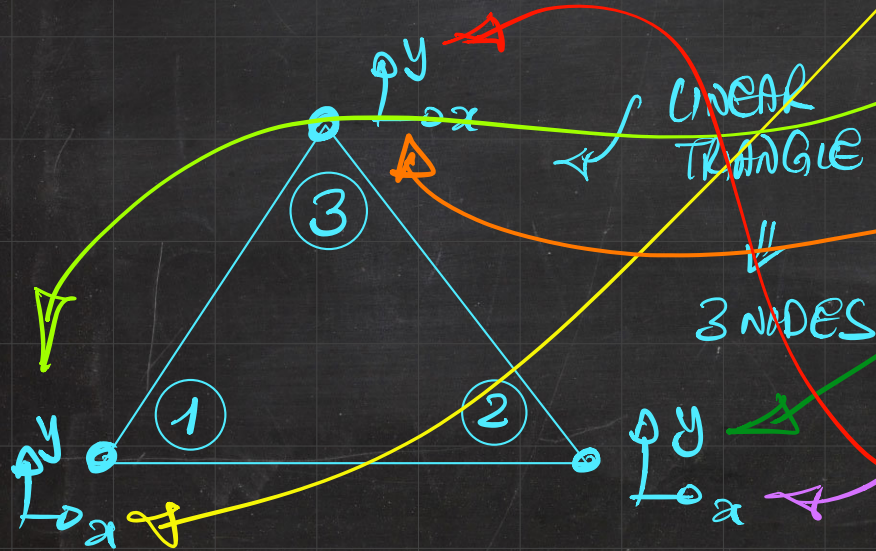


$$\begin{bmatrix} K_{11}^{11} & K_{12}^{11} & K_{11}^{12} & K_{12}^{12} & K_{11}^{13} & K_{12}^{13} \\ K_{12}^{11} & K_{22}^{11} & K_{12}^{12} & K_{22}^{12} & K_{12}^{13} & K_{22}^{13} \\ K_{11}^{12} & K_{12}^{12} & K_{11}^{21} & K_{12}^{21} & K_{11}^{23} & K_{12}^{23} \\ K_{12}^{12} & K_{22}^{12} & K_{12}^{21} & K_{22}^{21} & K_{12}^{23} & K_{22}^{23} \\ K_{11}^{13} & K_{12}^{13} & K_{11}^{23} & K_{12}^{23} & K_{11}^{31} & K_{12}^{31} \\ K_{12}^{13} & K_{22}^{13} & K_{12}^{23} & K_{22}^{23} & K_{12}^{31} & K_{22}^{31} \end{bmatrix}$$

$$\begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \\ u_1^3 \\ u_2^3 \end{bmatrix}$$

$$= F_{1 \rightarrow x}$$

FORCE OF
NODE 1
IN
DIRECTION 1
 x



FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD } w] : B \, dA = \int_{\partial B_N} w \cdot t_0 \, dL \quad [K]_{ac}^{ji} = \int_B [N]_{ac}^j E_{abcd} [N]_{cd}^i \, dA$$

$$[w^j]_a [K]_{ac}^{ji} [u^i]_c = [w^j]_q \int_{\partial B_N} n^j [t_0]_q \, dL$$

$$[K]_{2c}^{1i} [u^i]_c = [F^1]_2 \quad \sum_i \sum_c \dots$$

$$[\dots] \begin{bmatrix} \vdots \\ \vdots \end{bmatrix} = [F^1]_2$$

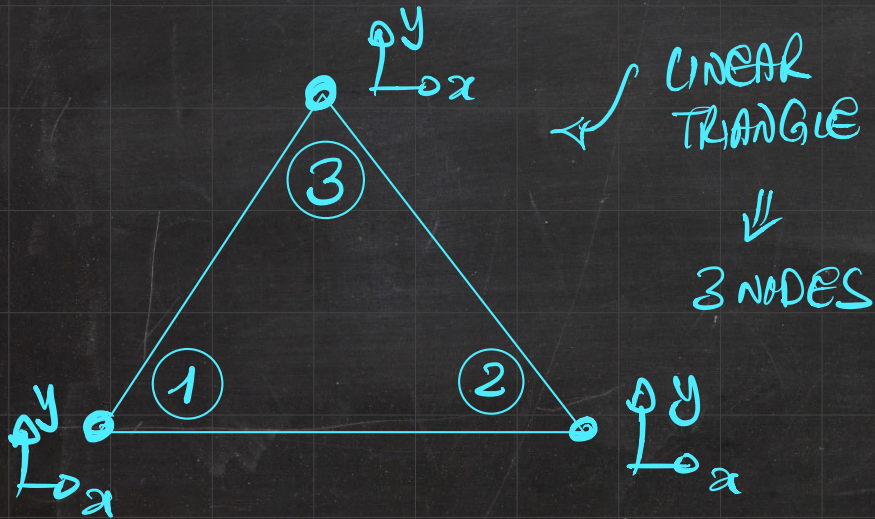
$$w^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$w^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$w^3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \dots$$

$$\begin{bmatrix} K_{21}^{11} & K_{22}^{11} & K_{21}^{12} & K_{22}^{12} & K_{21}^{13} & K_{22}^{13} \\ K_{21}^{21} & K_{22}^{21} & K_{21}^{22} & K_{22}^{22} & K_{21}^{23} & K_{22}^{23} \end{bmatrix}$$

$$\begin{bmatrix} u_1^1 \\ u_2^1 \\ u_1^2 \\ u_2^2 \\ u_1^3 \\ u_2^3 \end{bmatrix} = F_2^1$$



$$\begin{bmatrix}
 K_{11}^{11} & K_{12}^{11} & K_{11}^{12} & K_{12}^{12} & K_{11}^{13} & K_{12}^{13} \\
 K_{21}^{11} & K_{22}^{11} & K_{21}^{12} & K_{22}^{12} & K_{21}^{13} & K_{22}^{13}
 \end{bmatrix}
 \begin{bmatrix}
 u_1^1 \\
 u_2^1 \\
 u_1^2 \\
 u_2^2 \\
 u_1^3 \\
 u_2^3
 \end{bmatrix}
 = F_2^1$$

$[K]_{6 \times 6}$ ✓

$[K^{11}]_{2 \times 2}$

$$K^{11} = \begin{bmatrix} K_{xx}^{11} & K_{xy}^{11} \\ K_{yx}^{11} & K_{yy}^{11} \end{bmatrix}$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD} w] : \mathbf{B} \, dA = \int_{\partial B_N} w \cdot \mathbf{t}_0 \, dL \quad [K]_{ac}^{ji} = \int_B [N]_{12a}^j \mathbf{b} E_{abcd} [N]_{12a}^i \, dA$$

$$[w^j]_a [K]_{ac}^{ji} [u^i]_c = [w^j]_a \int_{\partial B_N} n^j [\mathbf{t}_0]_q \, dL$$

$$[K]_{ac}^{ji} [u^i]_c = [F]_a^j$$

$[K]_{ac}^{ji}$ $\rightarrow$ STIFFNESS
BETWEEN
NODES
 i & j
 $\hookrightarrow$ 2x2
matrix

FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD } w] : B \, dA = \int_{\partial B_N} w \cdot t_0 \, dL$$

$$[K^{ji}]_{ac} [u^i]_c = [F^j]_a$$

$$[K^{ji}]_{ac} = \int_B [N^j_{,a}]_b E_{abcd} [N^i_{,c}]_d \, dA$$

$[K^{ji}]_{ac}$ STIFFNESS BETWEEN NODES i & j
 ↳ 2x2 matrix

$$[K^{ji}] = \begin{bmatrix} K_{11}^{ji} & K_{12}^{ji} \\ K_{21}^{ji} & K_{22}^{ji} \end{bmatrix}$$

FROM WEAK FORM TO APPROXIMATE FORM

$$\int_B [\text{GRAD} w] : B \, dA = \int_{\partial B_N} w \cdot t_0 \, dL$$

$$[K^{ij}]_{ac} [u^j]_c = [F^i]_a$$

$$\{i, j\} \in \{1, 2, \dots, NPE\}$$

$$[K^{ij}]_{ac} = \int_B [N_{12a}]_b^i E_{abcd} [N_{12a}]_d^j \, dA$$

$[K^{ij}]$ $\swarrow$ STIFFNESS
BETWEEN
NODES
 i & j
 $\hookrightarrow$ 2×2
matrix

$$[K^{ij}] = \begin{bmatrix} K_{11}^{ij} & K_{12}^{ij} \\ K_{21}^{ij} & K_{22}^{ij} \end{bmatrix}$$