

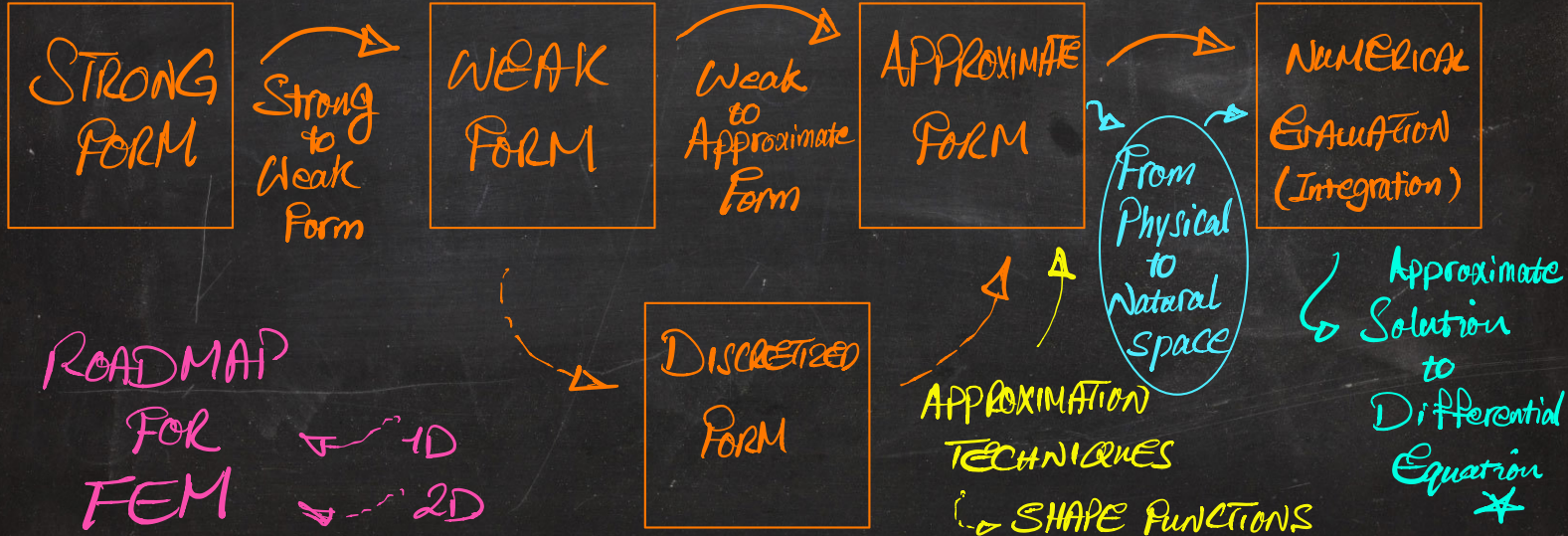
FINITE ELEMENT METHOD

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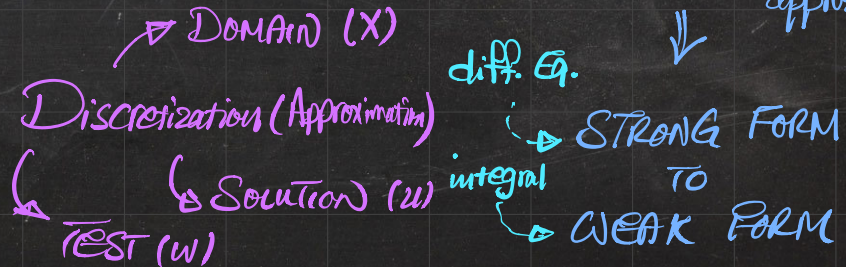
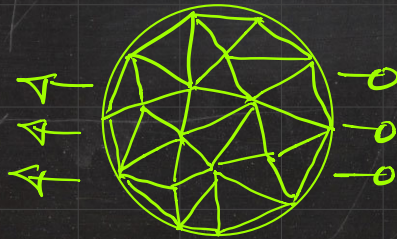
17

FINITE ELEMENT METHOD

Differential
Equation ★
↓



UNDERSTANDING FEM VIA AN ANALOGY (A BRUTAL SIMPLIFICATION)



$$\frac{d}{dx} \left(EA \frac{du}{dx} \right) + bA = 0 \quad \text{subject to BCs}$$

$$\hookrightarrow E, A: \text{Const.} \quad \rightarrow EA u'' + bA = 0 \quad \leftarrow f := \frac{b}{E}$$

STRONG
FORM

$$\rightarrow u'' + f = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = u_0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = t \quad \checkmark$$

FROM STRONG TO WEAK FORM

STRONG FORM $\leftarrow$ Differential Eq.

(I) MULTIPLY BY TEST FUNCTION w

(II) INTEGRATE OVER THE DOMAIN

Integral form $\leftarrow$

WEAK FORM

$\leftarrow$ BECAUSE LOWER ORDER DIFFERENTIATION OF DISPLACEMENT u

STRONG : u''

WEAK : u'

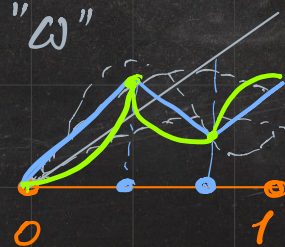
$$u'' + f = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = u_0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = t \quad \checkmark$$

$w : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ w|_D = 0 \quad \leftarrow \text{ZERO @} \end{cases}$

DIRICHLET BOUNDARY CONDITIONS



FROM STRONG TO WEAK FORM

$$u'' + f = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = u_0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = t \quad \checkmark$$

WEAK FORM

$$\int_0^1 \omega' u' dx = \int_0^1 \omega f dx + \underbrace{\omega(1) u'(1) - \omega(0) u'(0)}_{\text{EXTERNAL CONTRIBUTIONS OVER THE DOMAIN}}$$

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

INTERNAL

EXTERNAL

EXTERNAL CONTRIBUTIONS

CONTRIBUTIONS

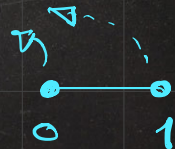
CONTRIBUTIONS

OVER THE BOUNDARY

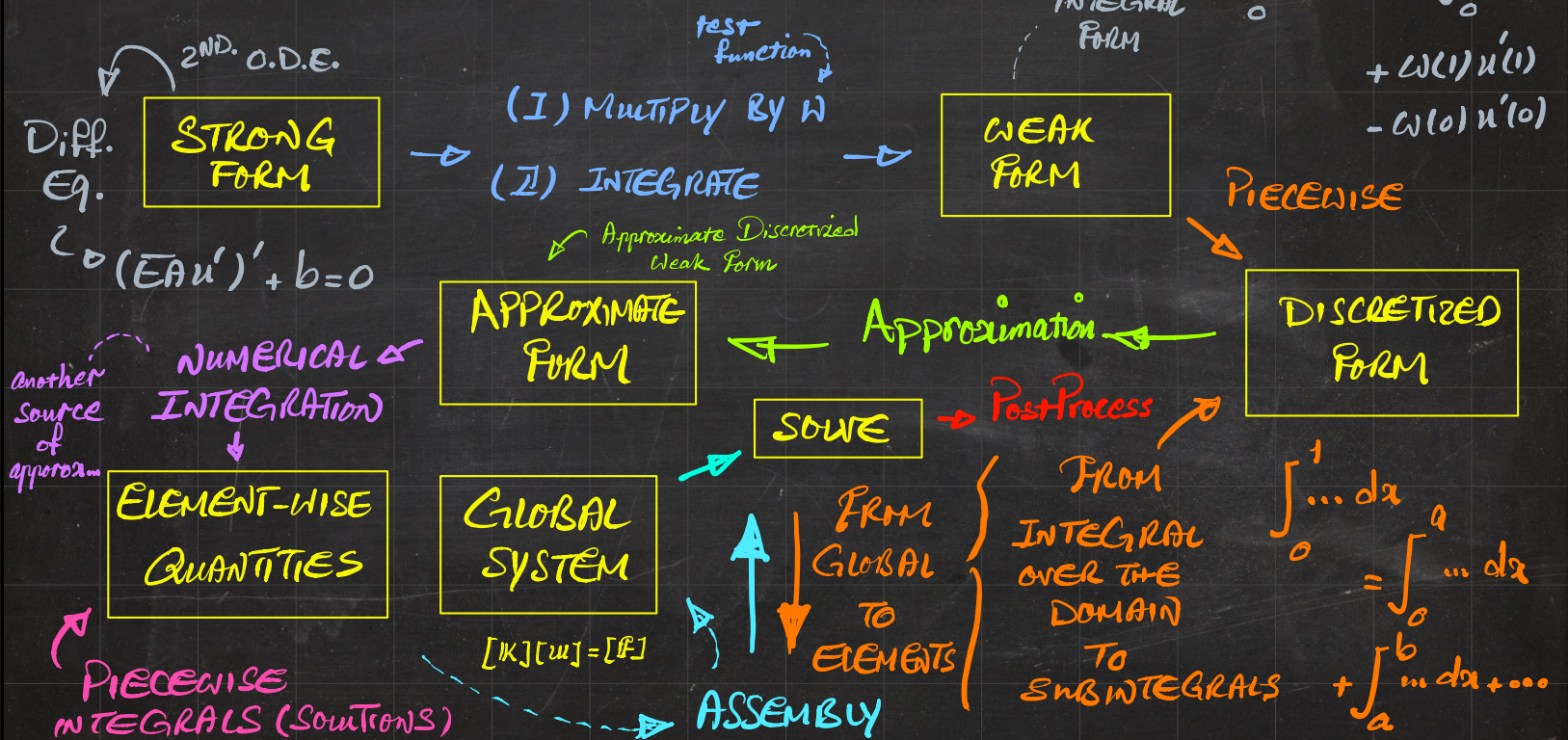
OVER THE DOMAIN

OVER THE DOMAIN

OF THE DOMAIN



FINITE ELEMENT METHOD IN A NUTSHELL

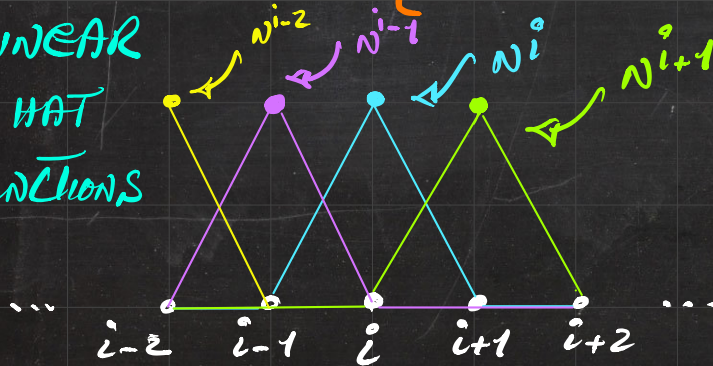


SHAPE FUNCTIONS (HAT FUNCTIONS, TENT FUNCTIONS)

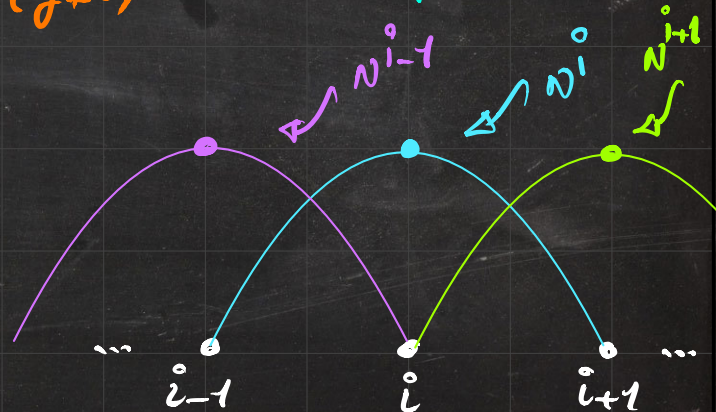
↳ A powerful tool for approximations $\Rightarrow$ SYSTEMATIC

$$N^i(x) \Rightarrow \begin{cases} N^i = 1 @ x^j (j=i) \\ N^i = 0 @ x^j (j \neq i) \end{cases} \Rightarrow \text{NEARLY IDENTICAL FOR 2D 3D}$$

LINEAR
HAT
FUNCTIONS



QUADRATIC HAT
FUNCTIONS



SHAPE FUNCTIONS (HAT FUNCTIONS, TENT FUNCTIONS)

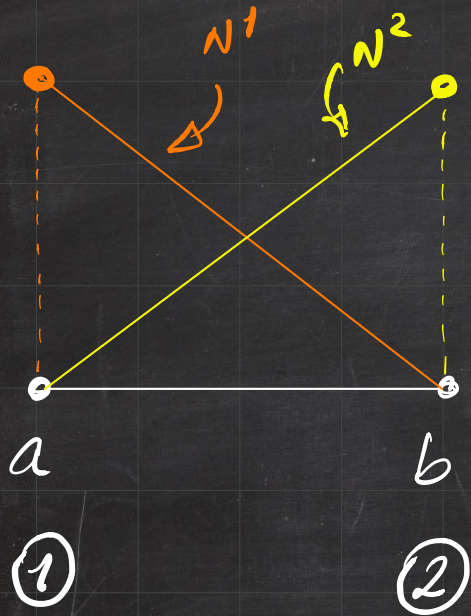
↳ A powerful tool for approximations $\rightarrow$ SYSTEMATIC

$$N^i(x) \rightarrow \begin{cases} N^i = 1 @ x^j (j=i) \\ N^i = 0 @ x^j (j \neq i) \end{cases} \rightarrow \text{NEARLY IDENTICAL FOR } \begin{matrix} 2D \\ 3D \end{matrix}$$

NODES
PER
ELEMENT $\rightarrow$
NPE

$$u \approx \sum_{i=1} N^i u^i$$

$$\rightarrow \begin{cases} u = N^1 u^1 + N^2 u^2 & \swarrow \text{quadratic} \\ u = N^1 u^1 + N^2 u^2 + N^3 u^3 & \swarrow \text{approximation} \\ u = N^1 u^1 + N^2 u^2 + N^3 u^3 + N^4 u^4 & \swarrow \text{cubic approximation} \end{cases}$$



$$N^1 = \frac{x-b}{a-b}$$

$$N^2 = \frac{x-a}{b-a}$$

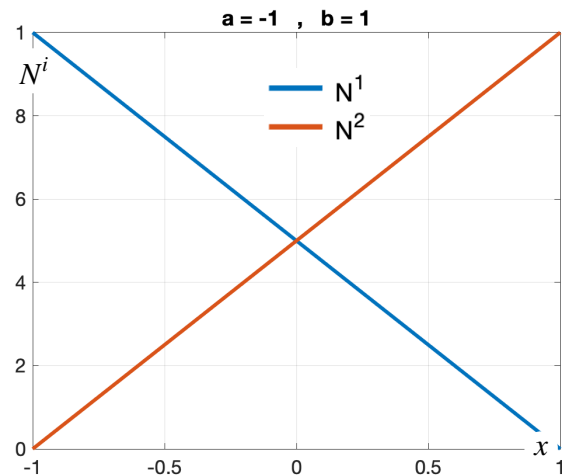
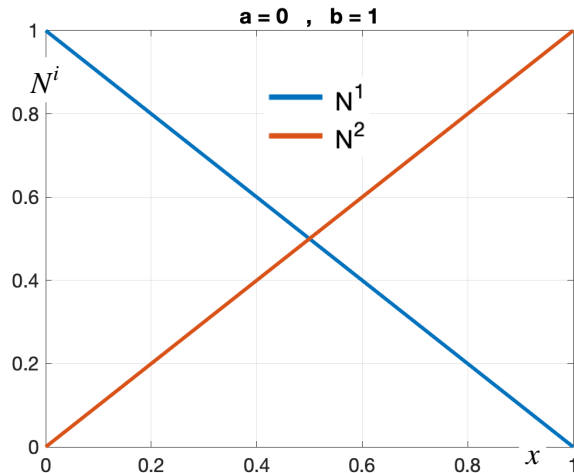
LINEAR
SHAPE
FUNCTIONS



1D Linear Shape Functions

$$N^1 = \frac{[x - b]}{[a - b]}$$

$$N^2 = \frac{[x - a]}{[b - a]}$$

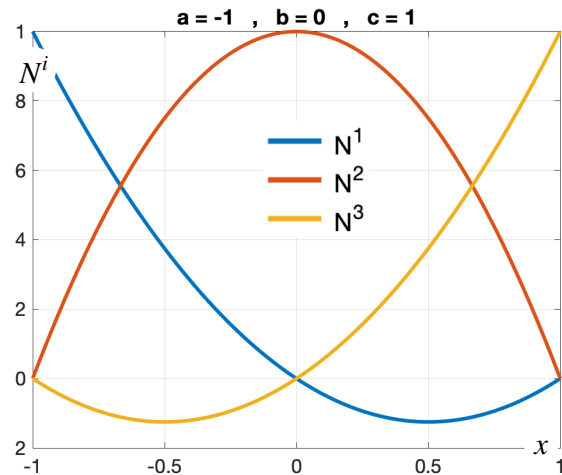
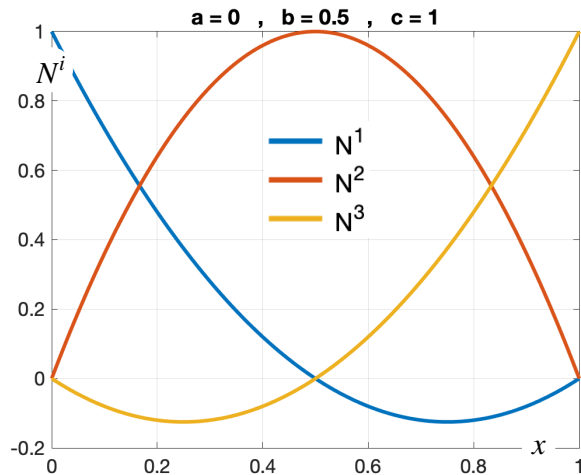


1D Quadratic Shape Functions

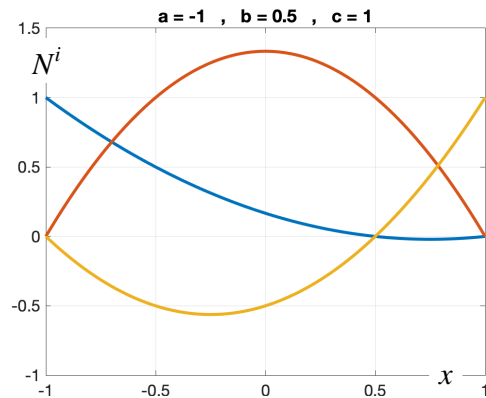
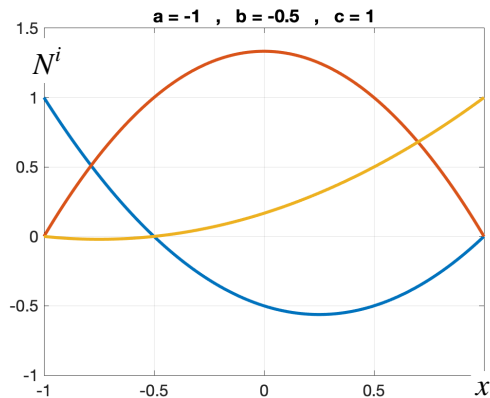
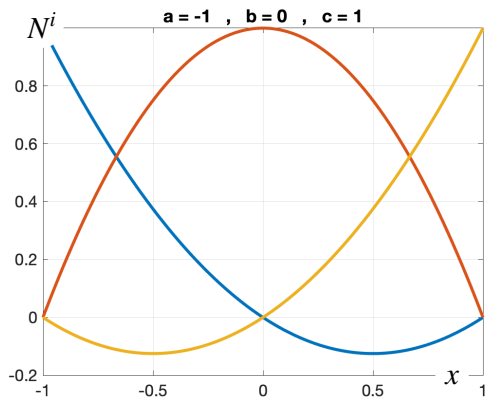
$$N^1 = \frac{[x - b][x - c]}{[a - b][a - c]}$$

$$N^2 = \frac{[x - a][x - c]}{[b - a][b - c]}$$

$$N^3 = \frac{[x - a][x - b]}{[c - a][c - b]}$$



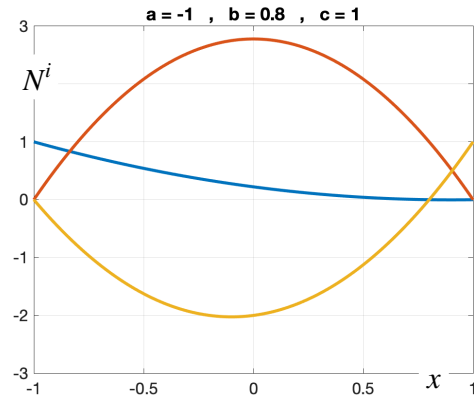
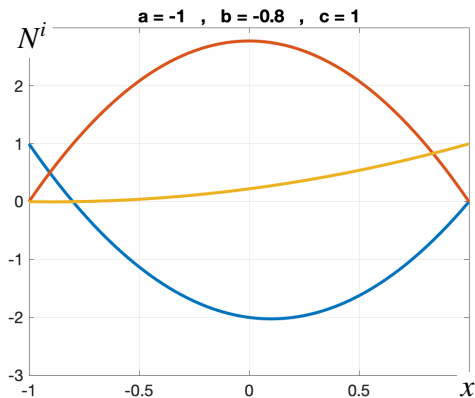
1D Quadratic Shape Functions



$$N^1 = \frac{[x-b][x-c]}{[a-b][a-c]}$$

$$N^2 = \frac{[x-a][x-c]}{[b-a][b-c]}$$

$$N^3 = \frac{[x-a][x-b]}{[c-a][c-b]}$$



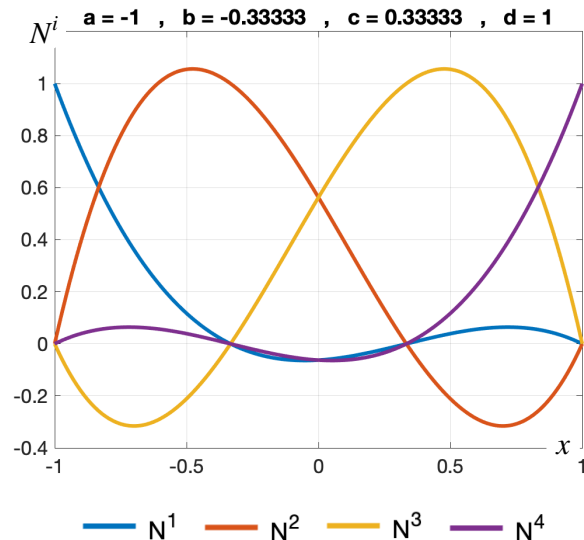
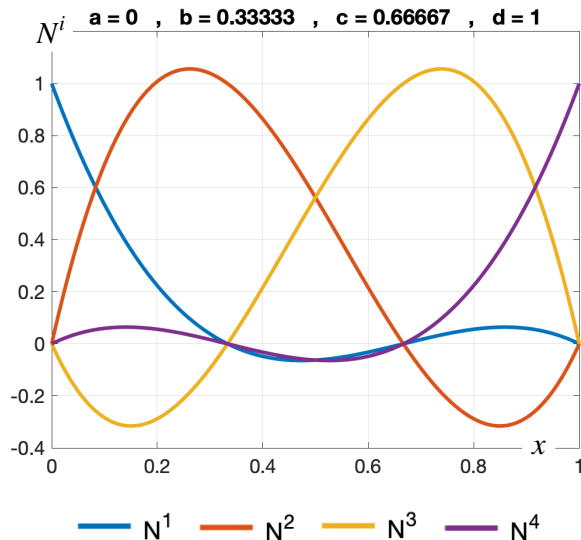
1D Cubic Shape Functions

$$N^1 = \frac{[x - b][x - c][x - d]}{[a - b][a - c][a - d]}$$

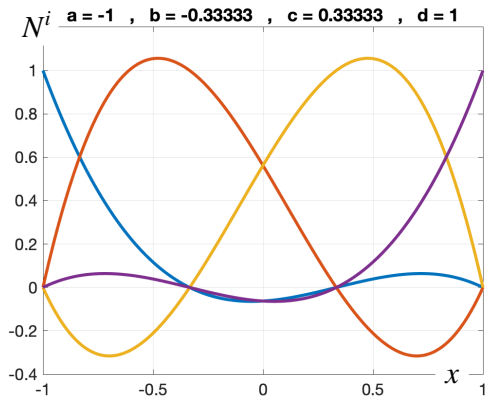
$$N^2 = \frac{[x - a][x - c][x - d]}{[b - a][b - c][b - d]}$$

$$N^3 = \frac{[x - a][x - b][x - d]}{[c - a][c - b][c - d]}$$

$$N^4 = \frac{[x - a][x - b][x - c]}{[d - a][d - b][d - c]}$$



1D Cubic Shape Functions

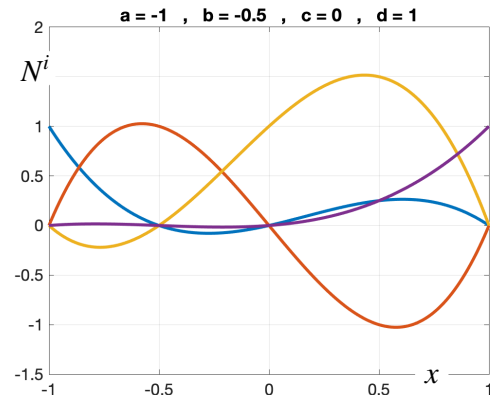
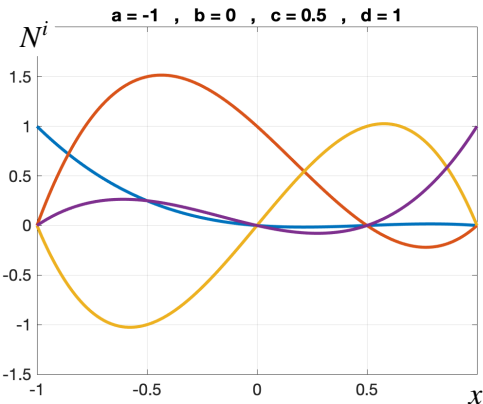
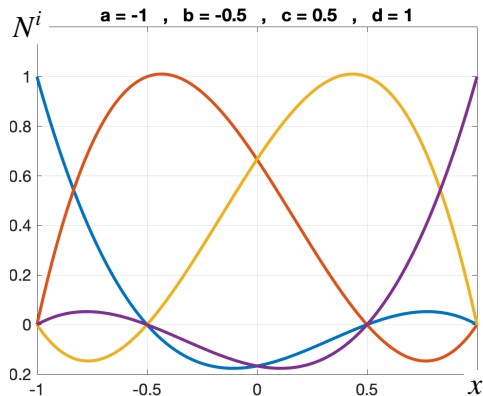
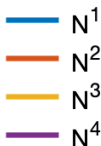


$$N^1 = \frac{[x - b][x - c][x - d]}{[a - b][a - c][a - d]}$$

$$N^2 = \frac{[x - a][x - c][x - d]}{[b - a][b - c][b - d]}$$

$$N^3 = \frac{[x - a][x - b][x - d]}{[c - a][c - b][c - d]}$$

$$N^4 = \frac{[x - a][x - b][x - c]}{[d - a][d - b][d - c]}$$



FROM STRONG TO WEAK FORM

$$u'' = -1 \Rightarrow u' = -x + C_1$$

$$\Rightarrow u = -\frac{1}{2}x^2 + C_1x + C_2$$

$$\hookrightarrow u(0) = 0 \Rightarrow C_2 = 0$$

$$\hookrightarrow u'(1) = 0 \Rightarrow C_1 = 1$$

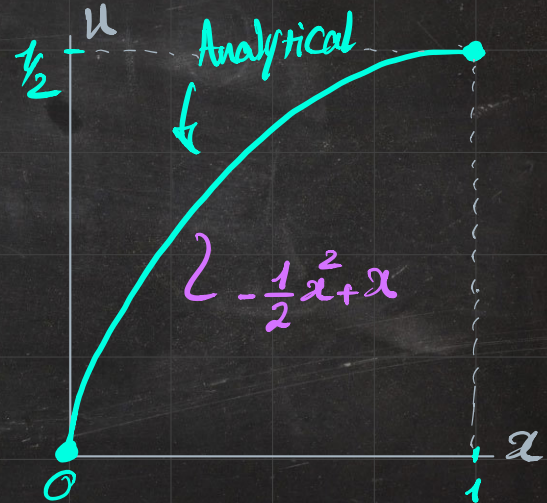
Analytical
Solution

$$\Rightarrow u = -\frac{1}{2}x^2 + x$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$



UNDERSTANDING THE SIGNIFICANCE OF

WEAK FORM

By Example $\int_0^1 \omega' u' dx = \int_0^1 \omega dx$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

→ 1-PIECE LINEAR APPROXIMATION

→ 2-PIECE LINEAR (UNIFORM) APPROXIMATION

→ 1-PIECE QUADRATIC APPROXIMATION

$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

REMEMBER HOW WE DID BEFORE

$$\omega = C_1 x + C_2 \quad u = D_1 x + D_2$$

$$\dots \int_0^1 \omega' u' dx = \int_0^1 \omega dx \dots \forall \omega$$

$$\dots \Rightarrow D_1 \& D_2 \checkmark$$

UNDERSTANDING THE SIGNIFICANCE OF

WEAK FORM

BY EXAMPLE

$$\int_0^1 \omega' u' dx = \int_0^1 \omega dx$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

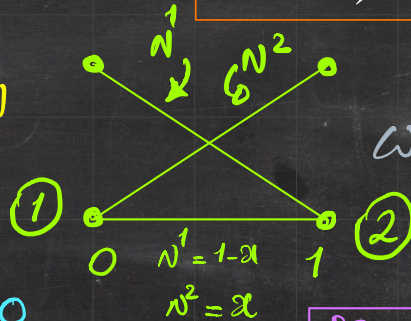
→ 1-PIECE LINEAR APPROXIMATION

$$\omega = N^1 \omega^1 + N^2 \omega^2$$

$$u = N^1 u^1 + N^2 u^2$$

$$\omega^1 = 0 \quad \omega|_D = 0$$

$$u^1 = 0 \quad u(0) = 0$$



$$\omega : \begin{cases} \text{ARBITRARY} \\ \text{CONTINUOUS} \\ \omega|_D = 0 \end{cases}$$

$$\omega = N^2 \omega^2 = x \omega^2$$

$$u = N^2 u^2 = x u^2 \Rightarrow u = \frac{1}{2} x \checkmark$$

$$\int_0^1 \omega^2 u^2 dx = \int_0^1 x \omega^2 dx \rightarrow \omega^2 u^2 = \omega^2 \frac{1}{2} \quad \sqrt{\omega^2} \Rightarrow u^2 = \frac{1}{2}$$

REMEMBER HOW WE DID BEFORE

$$\omega = C_1 x + C_2 \quad u = D_1 x + D_2$$

$$\dots \int_0^1 \omega' u' dx = \int_0^1 \omega dx \dots \forall \omega$$

$$\dots \Rightarrow D_1 \& D_2 \checkmark$$

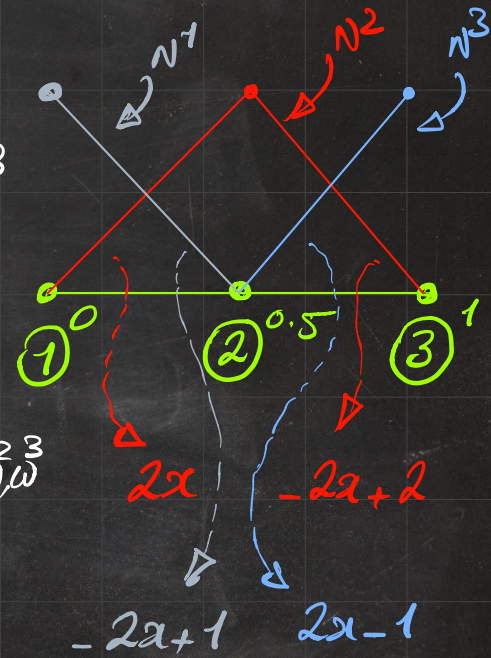
2. Piece LINEAR UNIFORM APPROXIMATION

$$w = N^1 \cancel{w^0} + N^2 w^2 + N^3 w^3$$

$$\Rightarrow w = N^2 w^2 + N^3 w^3$$

$$u = N^1 \cancel{u^0} + N^2 u^2 + N^3 u^3$$

$$\Rightarrow u = N^2 u^2 + N^3 u^3$$



$$w^2 \left[4u^2 - 2u^3 - \frac{1}{2} \right] + w^3 \left[2u^3 - 2u^2 - \frac{1}{4} \right] = 0 \quad \forall w^2, w^3$$

$$\begin{cases} 4u^2 - 2u^3 - \frac{1}{2} = 0 \\ 2u^3 - 2u^2 - \frac{1}{4} = 0 \end{cases} \Rightarrow \begin{cases} u^2 = \frac{3}{8} \\ u^3 = \frac{1}{2} \end{cases} \Rightarrow u = u(x) \quad \checkmark$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

1. Piece QUADRATIC APPROXIMATION

$$w = N^1 \cancel{w^1}^0 + N^2 w^2 + N^3 w^3$$

$$u = N^1 \cancel{u^1}^0 + N^2 u^2 + N^3 u^3$$

$$\begin{bmatrix} 16/3 & -8/3 \\ -8/3 & 16/3 \end{bmatrix} \begin{bmatrix} u^2 \\ u^3 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/6 \end{bmatrix}$$

$$\Rightarrow \begin{cases} u^2 = 3/8 \\ u^3 = 1/2 \end{cases}$$

$$\begin{aligned} u &= N^2 u^2 + N^3 u^3 \\ &= [-4x^2 + 4x] \cdot 3/8 \\ &\quad + [2x^2 - x] \cdot 1/2 \end{aligned}$$

$$\Rightarrow u = -\frac{1}{2}x^2 + x$$

this was analytical solution



$$N^1 = 2[x-0.5][x-1] = 2x^2 - 3x + 1$$

$$N^2 = -4[x-0][x-1] = -4x^2 + 4x$$

$$N^3 = 2[x-0][x-0.5] = 2x^2 - x$$

$$N^{1'} = 4x - 3$$

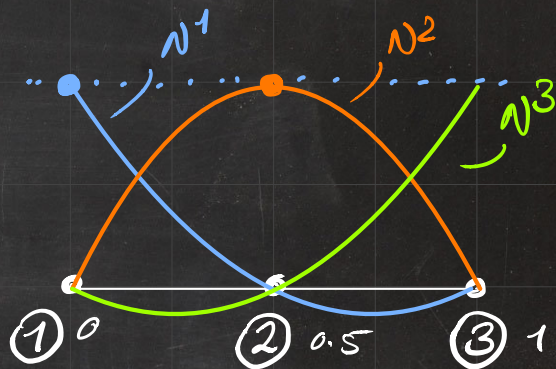
$$N^{2'} = -8x + 4$$

$$N^{3'} = 4x - 1$$

$$u'' + 1 = 0 \quad 0 \leq x \leq 1$$

$$D: u(0) = 0 \quad \leftarrow \text{prescribed}$$

$$N: u'(1) = 0 \quad \checkmark$$

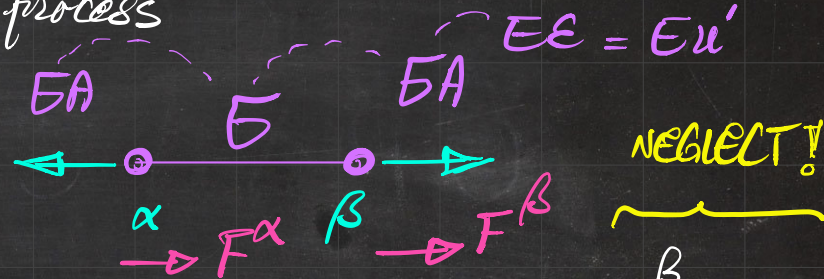


Consider the generic strong form $(EA u')' + b = 0 \quad \alpha \leq x \leq \beta$

↳ ... ↗ weak from derivation process

subject to BCs at α, β

Corresponding weak form



$$\int_{\alpha}^{\beta} EA u' u' dx = \underbrace{EA u'(\beta) u(\beta)}_{F^{\beta}} - \underbrace{EA u'(\alpha) u(\alpha)}_{F^{\alpha}} + \underbrace{\int_{\alpha}^{\beta} b u dx}_{\text{body forces over the domain}}$$

$F^{\alpha, \beta}$: EXTERNAL FORCES AT NODES α, β

Consider the generic strong form $(EA u')' + b = 0 \quad \alpha \leq x \leq \beta$

$\hookrightarrow \dots \rightarrow$ weak form
derivation process

subject to BCs at α, β

Corresponding weak form

$F^{\alpha, \beta}$: EXTERNAL
FORCES AT NODES
 α, β

$$EA \int_{\alpha}^{\beta} w' u' dx = w(\beta) F^{\beta} + w(\alpha) F^{\alpha}$$

Approximate using
 1- Piece linear
 1- Piece Quadratic

$$EA \int_{\alpha}^{\beta} w' u' dx = w(\beta) F^{\beta} + w(\alpha) F^{\alpha}$$

$$[K][u] = [F]$$

$$\hookrightarrow \begin{bmatrix} u^1 \\ u^2 \end{bmatrix} \hookrightarrow \begin{bmatrix} F^1 \\ F^2 \end{bmatrix}$$

$$K = EA$$

$$\begin{bmatrix} \int_{\alpha}^{\beta} N^1 N^1 dx & \int_{\alpha}^{\beta} N^1 N^2 dx \\ \int_{\alpha}^{\beta} N^2 N^1 dx & \int_{\alpha}^{\beta} N^2 N^2 dx \end{bmatrix}$$

$$\hookrightarrow K = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Approximate using
1-PIECE LINEAR

1-PIECE QUADRATIC

$$K = \frac{EA}{3L} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

$$[K][u] = [F]$$

$$\begin{bmatrix} u^1 \\ u^2 \\ u^3 \end{bmatrix} \quad \begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix}$$

$$EA \int_{\alpha}^{\beta} w' u' dx = w(\beta) F^{\beta} + w(\alpha) F^{\alpha}$$

$$K = EA \begin{bmatrix} \int_{\alpha}^{\beta} N^1 N^1 dx & \int_{\alpha}^{\beta} N^1 N^2 dx & \int_{\alpha}^{\beta} N^1 N^3 dx \\ \int_{\alpha}^{\beta} N^2 N^1 dx & \int_{\alpha}^{\beta} N^2 N^2 dx & \int_{\alpha}^{\beta} N^2 N^3 dx \\ \int_{\alpha}^{\beta} N^3 N^1 dx & \int_{\alpha}^{\beta} N^3 N^2 dx & \int_{\alpha}^{\beta} N^3 N^3 dx \end{bmatrix}$$

GENERAL STRUCTURE OF STIFFNESS MATRIX FOR 1D FINITE ELEMENTS

$$K = EA \left[\begin{matrix} \text{NPE} \times \text{NPE} \end{matrix} \right]$$

NPE: Node Per Element

[NPE x PD] x [NPE x PD]

	1D	2D
LINEAR TRUSS	2x2	4x4
QUADR. TRUSS	3x3	6x6

$$K = EA \begin{bmatrix} \int_{\alpha}^{\beta} N^1{}' N^1{}' d\alpha & \int_{\alpha}^{\beta} N^1{}' N^2{}' d\alpha \\ \int_{\alpha}^{\beta} N^2{}' N^1{}' d\alpha & \int_{\alpha}^{\beta} N^2{}' N^2{}' d\alpha \end{bmatrix}$$

LINEAR TRUSS ELEMENT

$\Rightarrow K^{ij} = EA \int_{\alpha}^{\beta} N^i{}' N^j{}' d\alpha$

GENERAL STRUCTURE OF STIFFNESS MATRIX FOR 1D FINITE ELEMENTS

$$K = EA \left[\begin{matrix} \text{NPE} \times \text{NPE} \end{matrix} \right]$$

NPE: Node Per Element

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	1D	2D
LINEAR TRUSS	2x2	4x4
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$$K = EA \begin{bmatrix} \int_{\alpha}^{\beta} N^1 N^1 dx & \int_{\alpha}^{\beta} N^1 N^2 dx & \int_{\alpha}^{\beta} N^1 N^3 dx \\ \int_{\alpha}^{\beta} N^2 N^1 dx & \int_{\alpha}^{\beta} N^2 N^2 dx & \int_{\alpha}^{\beta} N^2 N^3 dx \\ \int_{\alpha}^{\beta} N^3 N^1 dx & \int_{\alpha}^{\beta} N^3 N^2 dx & \int_{\alpha}^{\beta} N^3 N^3 dx \end{bmatrix}$$

QUADRATIC TRUSS ELEMENT

$\hookrightarrow K^{ij} = EA \int_{\alpha}^{\beta} N^i N^j dx$

GENERAL STRUCTURE OF STIFFNESS MATRIX FOR 1D FINITE ELEMENTS

$$K = EA \begin{bmatrix} NPE \times NPE \end{bmatrix}$$

NPE: Node Per Element

[NPE x PD] x [NPE x PD]

	1D	2D
LINEAR	2x2	4x4
TRUSS		
QUADR. TRUSS	3x3	6x6

$K_{ij} = EA \int_{\alpha}^{\beta} N_i' N_j' dx$

$N_i' = N_i'(x)$ $N_j' = N_j'(x)$

$N_i' = N_i'(x)$ $N_j' = N_j'(x)$

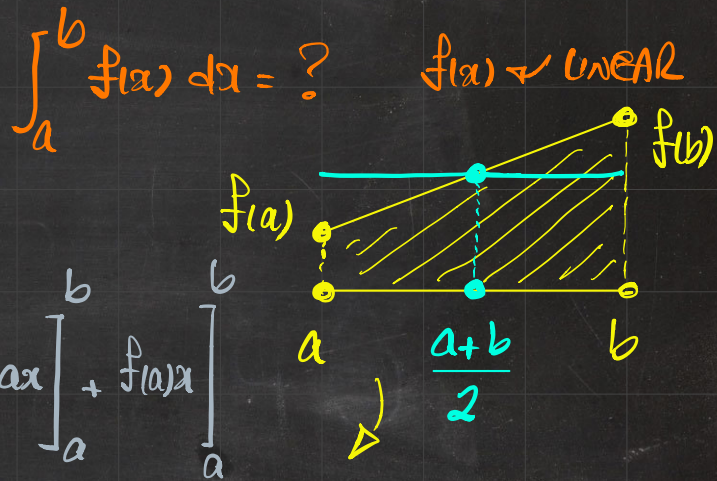
$= EA \int_{\alpha}^{\beta} f(x) dx$

EVALUATE THIS INTEGRAL

NUMERICALLY

NUMERICAL INTEGRATION :

$$f(x) = \frac{f(b)-f(a)}{b-a} [x-a] + f(a)$$



$$\int_a^b f(x) dx = \frac{f(b)-f(a)}{b-a} \left[\frac{1}{2} x^2 \right]_a^b - \frac{f(b)-f(a)}{b-a} ax \Big|_a^b + f(a)x \Big|_a^b$$

$$= \underbrace{\frac{f(b)+f(a)}{2}}_{f\left(\frac{a+b}{2}\right)} \underbrace{[b-a]}_L$$

$$A = \frac{1}{2} [f(a) + f(b)] [b-a]$$

NUMERICAL INTEGRATION :

$$\int_a^b f(x) dx = ? \quad f(x) \nrightarrow \text{LINEAR}$$

$$\int_a^b f(x) dx = [b-a] f\left(\frac{a+b}{2}\right)$$

EXACT
INTEGRAL

LENGTH

EVALUATION
OF $f(x)$
ONLY AT
ONE
POINT

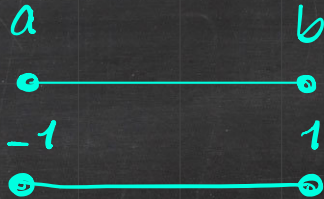
THIS IS
EXACT

(not even an
approximation)

NUMERICAL INTEGRATION :

$$\int_a^b f(x) dx = ? \quad f(x) \leftarrow \text{QUADRATIC}$$

$$\underbrace{\int_a^b f(x) dx = \dots}_{\text{EXACT INTEGRAL}}$$



EXACT
INTEGRAL

ξ

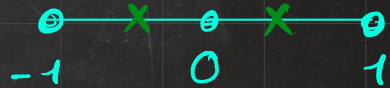


$$\int_{-1}^1 g(\xi) d\xi = ?$$

$$g(\xi) = C_1 \xi^2 + C_2 \xi + C_3$$

NUMERICAL INTEGRATION :

$$g(\xi) = C_1 \xi^2 + C_2 \xi + C_3$$



$$\int_{-1}^1 g(\xi) d\xi = \frac{2}{3} C_1 + 2 C_3 \pm \frac{1}{\sqrt{3}} C_2$$

$$= \left[\frac{1}{3} C_1 + \frac{1}{\sqrt{3}} C_2 + C_3 \right] + \left[\frac{1}{3} C_1 - \frac{1}{\sqrt{3}} C_2 + C_3 \right]$$

$$= g\left(\frac{1}{\sqrt{3}}\right) + g\left(-\frac{1}{\sqrt{3}}\right)$$

NUMERICAL INTEGRATION :

↖ GAUSS QUADRATURE FORMULA
↳ GAUSS POINTS

$$\int_{-1}^1 g(\xi) d\xi = 2 \times g(0)$$

g : LINEAR

$$\int_{-1}^1 g(\xi) d\xi = g\left(\frac{1}{\sqrt{3}}\right) + g\left(-\frac{1}{\sqrt{3}}\right)$$

g : QUADRATIC

$$\int_{-1}^1 g(\xi) d\xi = g\left(\frac{1}{\sqrt{3}}\right) + g\left(-\frac{1}{\sqrt{3}}\right)$$

g : CUBIC

NUMERICAL INTEGRATION : $\leftarrow$ QUADRATURE RULE $m \rightarrow$ GAUSS POINTS

$$\int_{-1}^1 g(\xi) d\xi = \sum_{i=1}^{GPE} \alpha_i g(\xi_i)$$

$GPE \leftarrow$ NUMBER OF GAUSS POINTS PER ELEMENT
 α_i weight factor
 $GPE \geq \frac{P+1}{2}$ ORDER OF POLYNOMIAL

	P	GPE		
LINEAR	1	1	$m \rightarrow \xi_1 = 0, \alpha_1 = 2$	
QUADRATIC	2	2	$m \rightarrow \begin{cases} \xi_1 = -1/\sqrt{3} \\ \xi_2 = 1/\sqrt{3} \end{cases}$	$GPE \sum_{i=1}^2 \alpha_i = 2$
CUBIC	3	2		
4th O.	4	3	$n \rightarrow \begin{cases} \xi_1 = -\sqrt{0.6} \\ \xi_2 = 0 \\ \xi_3 = \sqrt{0.6} \end{cases}$	$\alpha_1 = 5/9, \alpha_2 = 8/9, \alpha_3 = 5/9$
5th O.	5	3		

NUMERICAL INTEGRATION :

$$\int_{-1}^1 g(\xi) d\xi = \sum_{i=1}^{GPE} \alpha_i g(\xi_i)$$

$$\int_a^b f(x) dx = ? \quad \swarrow \text{TRANSFORM TO} \quad \int_{-1}^1 g(\xi) d\xi \quad \nearrow$$

x -DOMAIN $\swarrow$ PHYSICAL SPACE

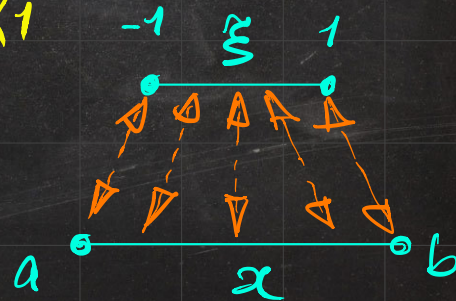
ξ -DOMAIN $\swarrow$ NATURAL SPACE

$$a \leq x \leq b$$

$$-1 \leq \xi \leq 1$$

MAPPING FROM PHYSICAL TO NATURAL SPACE

$$\hookrightarrow x \doteq x(\xi) \Rightarrow f(x) = f(x(\xi)) = g(\xi)$$



NUMERICAL INTEGRATION :

$$\int_{-1}^1 g(\xi) d\xi = \sum_{i=1}^{GPE} \alpha_i g(\xi_i)$$

$$\int_a^b f(x) dx = ? \quad \swarrow \text{TRANSFORM TO} \quad \int_{-1}^1 g(\xi) d\xi$$

$J = \frac{\partial x}{\partial \xi}$
Jacobian

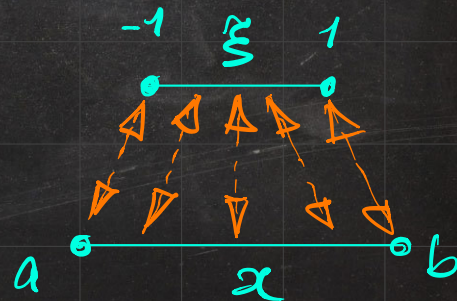
POLYNOMIAL ORDER

$$x \stackrel{?}{=} x(\xi) \Rightarrow x = \sum_{i=1}^{P+1} N^i x^i \quad \dots \quad dx = J(\xi) d\xi$$

$N^i = N^i(\xi)$

Shape Functions $\mapsto u = \sum N^i u^i$ values

$u(x) \quad N^i(x)$



NUMERICAL INTEGRATION :

$$\int_{-1}^1 g(\xi) d\xi = \sum_{i=1}^{GPE} \alpha_i g(\xi_i)$$

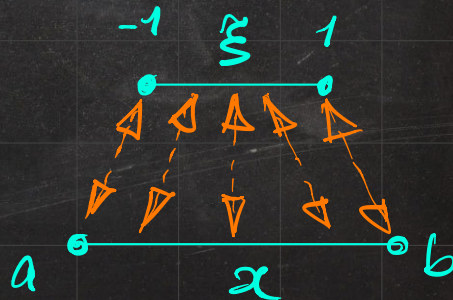
$$\int_a^b f(x) dx = ? \quad \swarrow \text{TRANSFORM TO} \quad \int_{-1}^1 g(\xi) d\xi$$

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$$x \stackrel{?}{=} x(\xi) \Rightarrow x = \sum_{i=1}^{P+1} N^i x^i \quad \dots \quad dx = J(\xi) d\xi$$

POLYNOMIAL ORDER
 $N^i = N^i(\xi)$

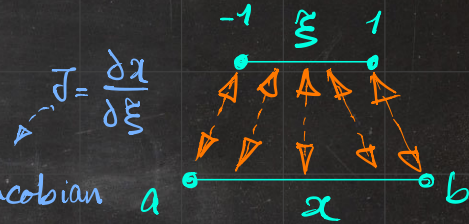
$$\int_a^b f(x) dx = \int_{-1}^1 \underbrace{f(x(\xi))}_{g(\xi)} J(\xi) d\xi = \sum_{i=1}^{GPE} \dots$$



NUMERICAL INTEGRATION :

$$\int_a^b f(x) dx = \int_{-1}^1 \underbrace{f(x(\xi))}_{g(\xi)} J(\xi) d\xi = \sum_{i=1}^{GPE} \dots$$

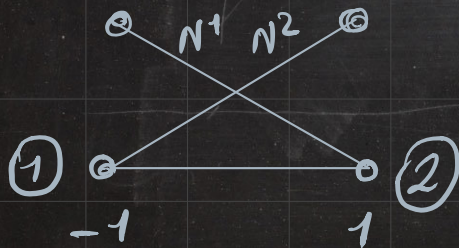
$$\int_{-1}^1 g(\xi) d\xi = \sum_{i=1}^{GPE} \alpha_i g(\xi_i)$$



Polynomial order $P+1$

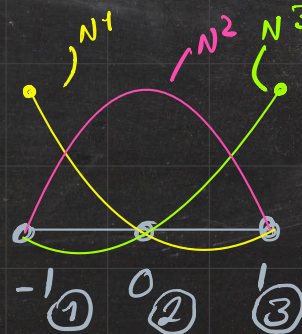
$$x = x(\xi) \Rightarrow x = \sum_{i=1}^{P+1} N^i x^i \dots dx = J(\xi) d\xi$$

$N^i = N^i(\xi)$



$$N^1 = -\frac{1}{2} [\xi - 1]$$

$$N^2 = \frac{1}{2} [\xi + 1]$$



$$N^1 = \frac{1}{2} \xi [\xi - 1]$$

$$N^2 = [1 - \xi][1 + \xi]$$

$$N^3 = \frac{1}{2} \xi [\xi + 1]$$

NUMERICAL INTEGRATION :

$$\int_{-1}^1 g(\xi) d\xi = \sum_{i=1}^{GPE} \alpha_i g(\xi_i)$$

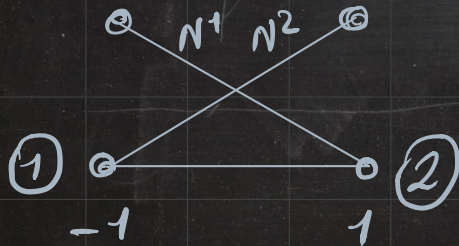
$$\int_a^b f(x) dx = \int_{-1}^1 \underbrace{f(x(\xi))}_{g(\xi)} J(\xi) d\xi = \sum_{i=1}^{GPE} \dots$$

$$J = \frac{\partial x}{\partial \xi}$$

Jacobian

$$x = x(\xi) \Rightarrow x = \sum_{i=1}^{P+1} N^i x^i \dots dx = J(\xi) d\xi$$

POLYNOMIAL ORDER
($N^i = N^i(\xi)$)



$$N^1 = -\frac{1}{2} [\xi - 1]$$

$$N^2 = \frac{1}{2} [\xi + 1]$$

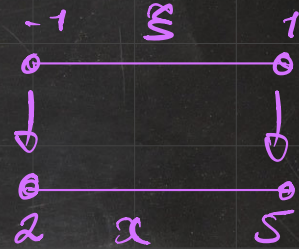
EXAMPLE: $\int_{-1}^1 [2x+1] dx = ?$

NUMERICAL INTEGRATION :

$$\int_a^b f(x) dx = \int_{-1}^1 \overbrace{[2[1.5\xi + 3.5] + 1] 1.5}^{g(\xi)} d\xi$$

$$= 2 \times g(0) = 24$$

$$\int_{-1}^1 g(\xi) d\xi = \sum_{i=1}^{GPE} \alpha_i g(\xi_i)$$



$$\alpha = \sum_{i=1}^2 N^i \alpha^i = N^1 \alpha^1 + N^2 \alpha^2 = -\frac{1}{2} [\xi - 1] \times 2 + \frac{1}{2} [\xi + 1] \times 5 = 1.5\xi + 3.5$$

$$\hookrightarrow \alpha = 1.5\xi + 3.5 \dots = x^2 + x$$

$$\hookrightarrow d\alpha = 1.5 d\xi$$

$$= 25 + 5 - 4 - 2 = 24$$

EXAMPLE :

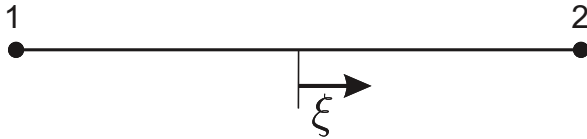
$$\int_2^5 [2\alpha + 1] d\alpha = ?$$

- one-dimensional 2-noded element (D1CU2N)
- one-dimensional 3-noded element (D1CU3N)
- one-dimensional 4-noded element (D1CU4N)
- one-dimensional 5-noded element (D1CU5N)
- one-dimensional 6-noded element (D1CU6N)
- one-dimensional 7-noded element (D1CU7N)
- one-dimensional quadrature rule
- one-dimensional constant body force nodal distribution

1D Finite Element Library

one-dimensional 2-noded element (D1CU2N)

Nodes



Node Number	Coordinate
	ξ
1	-1
2	1

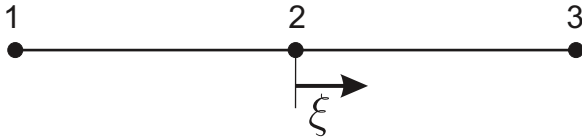
$$N^1 = -\frac{1}{2}(\xi - 1) \quad , \quad N^2 = \frac{1}{2}(\xi + 1)$$

$$N^1_{,\xi} = -\frac{1}{2} \quad , \quad N^2_{,\xi} = \frac{1}{2}$$

1D Finite Element Library

one-dimensional 3-noded element (D1CU3N)

Nodes



Node Number	Coordinate
	ξ
1	-1
2	0
3	1

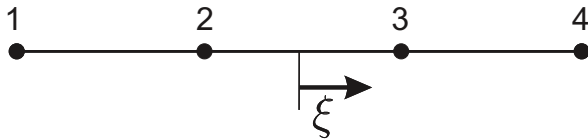
$$N^1 = \frac{1}{2}\xi(\xi - 1) \quad , \quad N^2 = -(\xi - 1)(\xi + 1) \quad , \quad N^3 = \frac{1}{2}\xi(\xi + 1)$$

$$N_{,\xi}^1 = \frac{1}{2}(2\xi - 1) \quad , \quad N_{,\xi}^2 = -2\xi \quad , \quad N_{,\xi}^3 = \frac{1}{2}(2\xi + 1)$$

1D Finite Element Library

one-dimensional 4-noded element (D1CU4N)

Nodes



Node Number	Coordinate
	ξ
1	-1
2	-1/3
3	1/3
4	1

$$N^1 = -\frac{9}{16} (\xi + 1/3) (\xi - 1/3) (\xi - 1)$$

$$N^2 = \frac{27}{16} (\xi + 1) (\xi - 1/3) (\xi - 1)$$

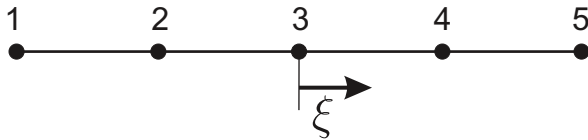
$$N^3 = -\frac{27}{16} (\xi + 1) (\xi + 1/3) (\xi - 1)$$

$$N^4 = \frac{9}{16} (\xi + 1) (\xi + 1/3) (\xi - 1/3)$$

1D Finite Element Library

one-dimensional 5-noded element (D1CU5N)

Nodes



Node Number	Coordinate
	ξ
1	-1
2	-1/2
3	0
4	1/2
5	1

$$N^1 = \frac{2}{3} (\xi + 1/2) \xi (\xi - 1/2) (\xi - 1)$$

$$N^2 = -\frac{8}{3} (\xi + 1) \xi (\xi - 1/2) (\xi - 1)$$

$$N^3 = 4 (\xi + 1) (\xi + 1/2) (\xi - 1/2) (\xi - 1)$$

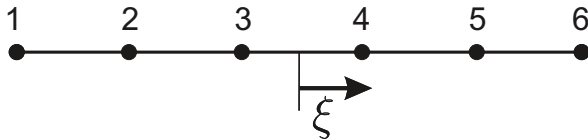
$$N^4 = -\frac{8}{3} (\xi + 1) (\xi + 1/2) \xi (\xi - 1)$$

$$N^5 = \frac{2}{3} (\xi + 1) (\xi + 1/2) \xi (\xi - 1/2)$$

1D Finite Element Library

one-dimensional 6-noded element (D1CU6N)

Nodes



Node Number	Coordinate
	ξ
1	-1
2	$-3/5$
3	$-1/5$
4	$1/5$
5	$3/5$
6	1

$$N^1 = -\frac{625}{768} (\xi + 3/5) (\xi + 1/5) (\xi - 1/5) (\xi - 3/5) (\xi - 1)$$

$$N^2 = \frac{3125}{768} (\xi + 1) (\xi + 1/5) (\xi - 1/5) (\xi - 3/5) (\xi - 1)$$

$$N^3 = -\frac{3125}{384} (\xi + 1) (\xi + 3/5) (\xi - 1/5) (\xi - 3/5) (\xi - 1)$$

$$N^4 = \frac{3125}{384} (\xi + 1) (\xi + 3/5) (\xi + 1/5) (\xi - 3/5) (\xi - 1)$$

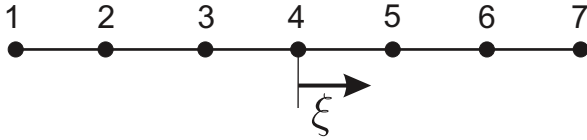
$$N^5 = -\frac{3125}{768} (\xi + 1) (\xi + 3/5) (\xi + 1/5) (\xi - 1/5) (\xi - 1)$$

$$N^6 = \frac{625}{768} (\xi + 1) (\xi + 3/5) (\xi + 1/5) (\xi - 1/5) (\xi - 3/5)$$

1D Finite Element Library

one-dimensional 7-noded element (D1CU7N)

Nodes



Node Number	Coordinate
	ξ
1	-1
2	$-2/3$
3	$-1/3$
4	0
5	$1/3$
6	$2/3$
7	1

$$N^1 = \frac{81}{80} (\xi + 2/3) (\xi + 1/3) \xi (\xi - 1/3) (\xi - 2/3) (\xi - 1)$$

$$N^2 = -\frac{243}{40} (\xi + 1) (\xi + 1/3) \xi (\xi - 1/3) (\xi - 2/3) (\xi - 1)$$

$$N^3 = \frac{243}{16} (\xi + 1) (\xi + 2/3) \xi (\xi - 1/3) (\xi - 2/3) (\xi - 1)$$

$$N^4 = -\frac{81}{4} (\xi + 1) (\xi + 2/3) (\xi + 1/3) (\xi - 1/3) (\xi - 2/3) (\xi - 1)$$

$$N^5 = \frac{243}{16} (\xi + 1) (\xi + 2/3) (\xi + 1/3) \xi (\xi - 2/3) (\xi - 1)$$

$$N^6 = -\frac{243}{40} (\xi + 1) (\xi + 2/3) (\xi + 1/3) \xi (\xi - 1/3) (\xi - 1)$$

$$N^7 = \frac{81}{80} (\xi + 1) (\xi + 2/3) (\xi + 1/3) \xi (\xi - 1/3) (\xi - 2/3)$$

one-dimensional quadrature rule (i)

$$\int_{-1}^1 \{\bullet\} d\xi \approx \sum_{i=1}^{N_{\text{Gauss Points}}} \alpha_i \{\bullet\}|_{\text{Gauss Point } i}$$

Gauss Point Number	Coordinate	Weight Factor
	ξ	α
1	0	2

Gauss Point Number	Coordinate	Weight Factor
	ξ	α
1	$-1/\sqrt{3}$	1
2	$1/\sqrt{3}$	1

Gauss Point Number	Coordinate	Weight Factor
	ξ	α
1	$-\sqrt{0.6}$	5/9
2	0	8/9
3	$\sqrt{0.6}$	5/9

1D Finite Element Library

one-dimensional quadrature rule (ii)

Gauss Point Number	Coordinate	Weight Factor
	ξ	α
1	-0.861 136 311 594 953	0.347 854 845 137 454
2	-0.339 981 043 584 856	0.652 145 154 862 546
3	0.339 981 043 584 856	0.652 145 154 862 546
4	0.861 136 311 594 953	0.347 854 845 137 454

Gauss Point Number	Coordinate	Weight Factor
	ξ	α
1	-0.906 179 845 938 664	0.236 926 885 056 189
2	-0.538 469 310 105 683	0.478 628 670 499 366
3	0.000 000 000 000 000	0.568 888 888 888 889
4	0.538 469 310 105 683	0.478 628 670 499 366
5	0.906 179 845 938 664	0.236 926 885 056 189

Gauss Point Number	Coordinate	Weight Factor
	ξ	α
1	-0.932 469 514 203 152	0.171 324 492 379 170
2	-0.661 209 386 466 265	0.360 761 573 048 139
3	-0.238 619 186 083 197	0.467 913 934 572 691
4	0.238 619 186 083 197	0.467 913 934 572 691
5	0.661 209 386 466 265	0.360 761 573 048 139
6	0.932 469 514 203 152	0.171 324 492 379 170

one-dimensional quadrature rule (iii)

Gauss Point Number	Coordinate	Weight Factor
	ξ	α
1	-0.949 107 912 342 759	0.129 484 966 168 870
2	-0.741 531 185 599 394	0.279 705 391 489 277
3	-0.405 845 151 377 397	0.381 830 050 505 119
4	0.000 000 000 000 000	0.417 959 183 673 469
5	0.405 845 151 377 397	0.381 830 050 505 119
6	0.741 531 185 599 394	0.279 705 391 489 277
7	0.949 107 912 342 759	0.129 484 966 168 870

1D Finite Element Library

one-dimensional constant body force nodal distribution

$$f^i = \frac{\int_{-1}^1 N^i d\xi}{\int_{-1}^1 d\xi}$$

Node number	1	2
Weight factor	1/2	1/2

Node number	1	2	3
Weight factor	1/6	4/6	1/6

Node number	1	2	3	4
Weight factor	1/8	3/8	3/8	1/8

Node number	1	2	3	4	5
Weight factor	7/90	32/90	12/90	32/90	7/90

Node number	1	2	3	4	5	6
Weight factor	19/288	25/96	25/144	25/144	25/96	19/288

Node number	1	2	3	4	5	6	7
Weight factor	41/840	18/70	9/280	68/210	9/280	18/70	41/840

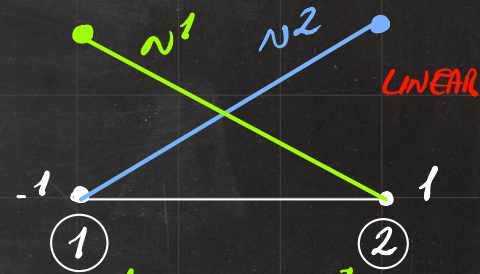
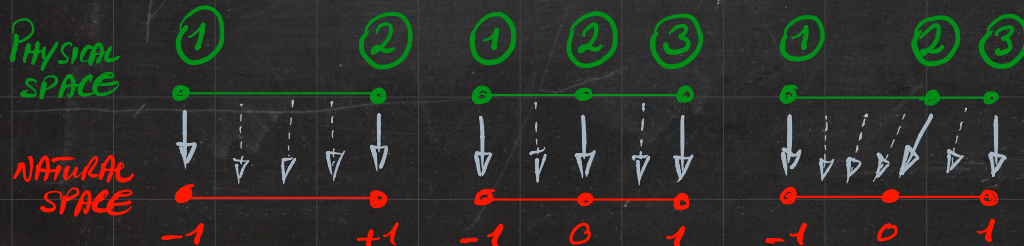
EXAMPLE: $\int_2^5 2x^2 + x - 1 dx = ?$

* ANALYTICAL SOLUTION

* APPROXIMATE SOLUTION $n \rightarrow$ LINEAR

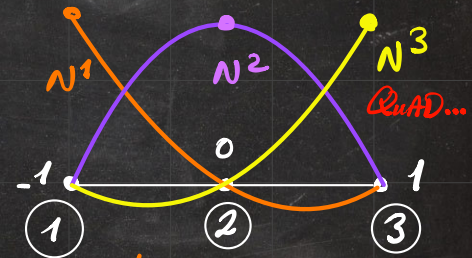
* APPROXIMATE SOLUTION $n \rightarrow$ QUADRATIC uniform

* APPROXIMATE SOLUTION $n \rightarrow$ QUADRATIC Non-uniform



$$N^1 = -\frac{1}{2}[\xi - 1]$$

$$N^2 = \frac{1}{2}[\xi + 1]$$



$$N^1 = \frac{1}{2}\xi[\xi - 1]$$

$$N^2 = [1 - \xi][1 + \xi]$$

$$N^3 = \frac{1}{2}\xi[\xi + 1]$$

EXAMPLE: $\int_2^5 2x^2 + x - 1 dx = ?$

↳ QUADRATIC Poly. IN x -DOMAIN

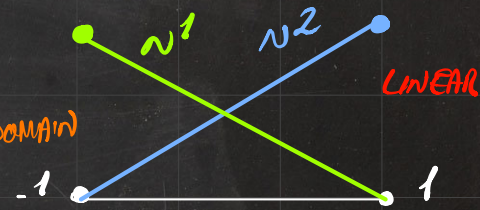
↳ GPE IS DEPENDENT ON POLYNOMIAL ORDER IN ξ -DOMAIN

* ANALYTICAL SOLUTION

* APPROXIMATE SOLUTION $n \rightarrow$ LINEAR

* APPROXIMATE SOLUTION $n \rightarrow$ QUADRATIC UNIFORM

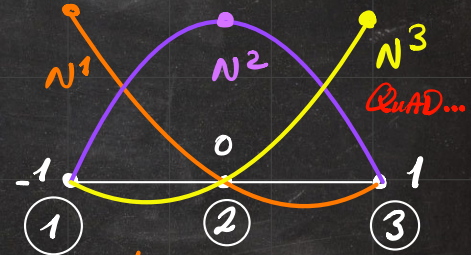
* APPROXIMATE SOLUTION $n \rightarrow$ QUADRATIC Non-uniform



(1) (2)

$$N^1 = -\frac{1}{2}[\xi - 1]$$

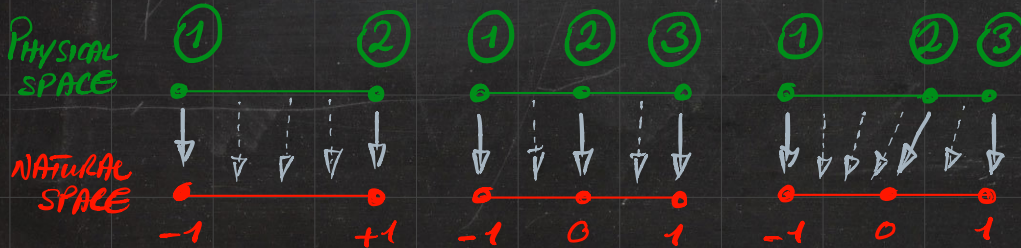
$$N^2 = \frac{1}{2}[\xi + 1]$$



$$N^1 = \frac{1}{2}\xi[\xi - 1]$$

$$N^2 = [1 - \xi][1 + \xi]$$

$$N^3 = \frac{1}{2}\xi[\xi + 1]$$



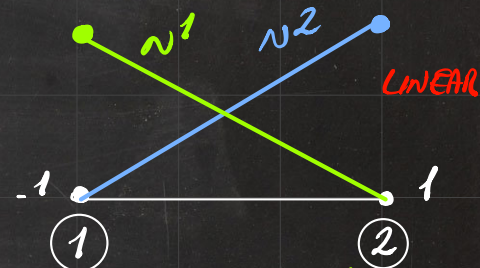
EXAMPLE: $\int_2^5 2x^2 + x - 1 dx = ?$

* ANALYTICAL SOLUTION

* APPROXIMATE SOLUTION $n \rightarrow$ LINEAR

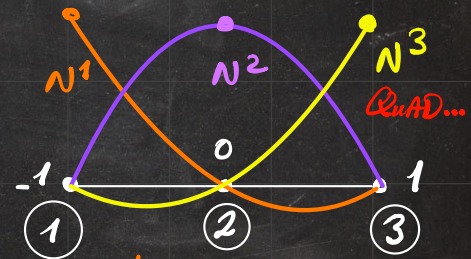
* APPROXIMATE SOLUTION $n \rightarrow$ QUADRATIC uniform

* APPROXIMATE SOLUTION $n \rightarrow$ QUADRATIC Non-uniform



$$N^1 = -\frac{1}{2}[\xi - 1]$$

$$N^2 = \frac{1}{2}[\xi + 1]$$



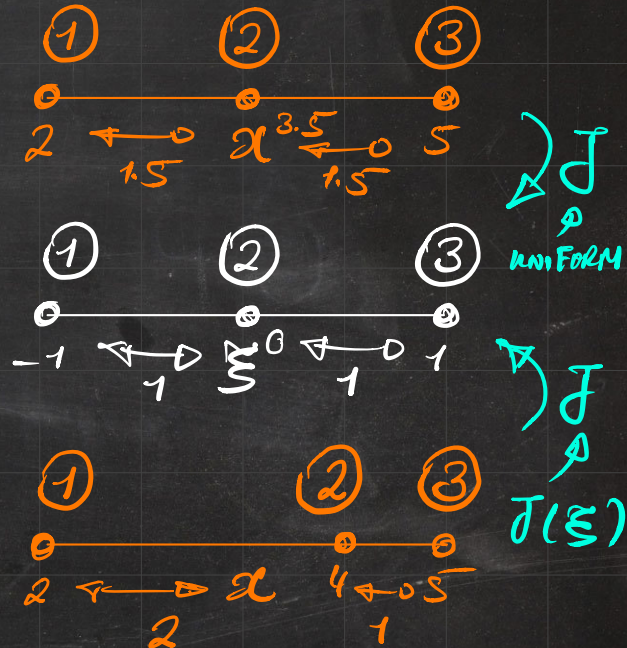
$$N^1 = \frac{1}{2}\xi[\xi - 1]$$

$$N^2 = [1 - \xi][1 + \xi]$$

$$N^3 = \frac{1}{2}\xi[\xi + 1]$$

NATURAL SPACE IS "GENERALLY" UNIFORMLY DISCRETIZED

THIS IS NOT THE CASE FOR
PHYSICAL SPACE NECESSARILY

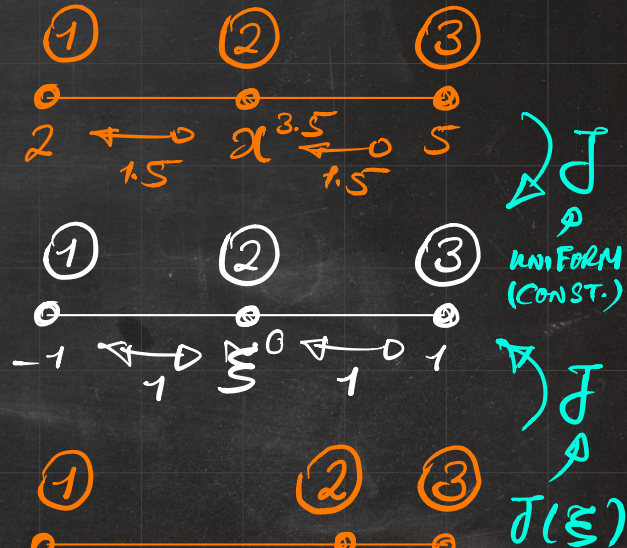


$$x = x(\xi) \Rightarrow dx = \frac{\partial x}{\partial \xi} d\xi$$

$$\int_a^b f(x) dx = \int_{-1}^1 \underbrace{f(x(\xi))}_{g(\xi)} J d\xi \Rightarrow \int_{-1}^1 g(\xi) d\xi$$

NATURAL SPACE IS "GENERALLY" UNIFORMLY DISCRETIZED

THIS IS NOT THE CASE FOR
PHYSICAL SPACE NECESSARILY



J UNIFORM IS DESIRABLE!

J : CONSTANT $\rightarrow$ ORDER OF $f(x)$
AND $g(\xi)$ ARE THE SAME!

NATURAL SPACE IS "GENERALLY" UNIFORMLY DISCRETIZED

↳ THIS IS NOT THE CASE FOR

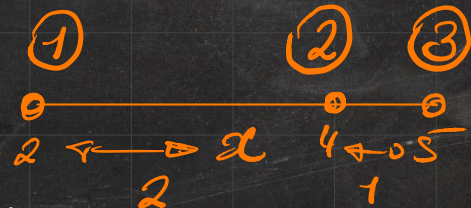
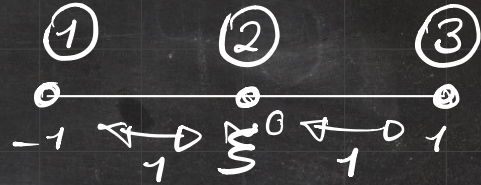
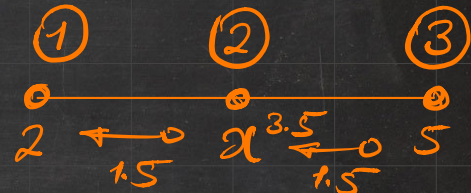
PHYSICAL SPACE NECESSARILY

↳ WHY IS IT NOT EASY?

↳ J UNIFORM IS DESIRABLE!



J : CONSTANT $\rightarrow$ ORDER OF $f(x)$
AND $g(\xi)$ ARE THE SAME!



J
 Δ
UNIFORM
(CONST.)

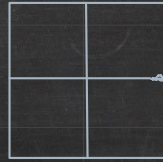
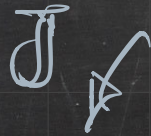
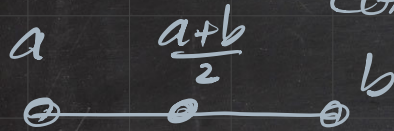
J
 Δ
 $J(\xi)$

WE WANT UNIFORM MAPPING

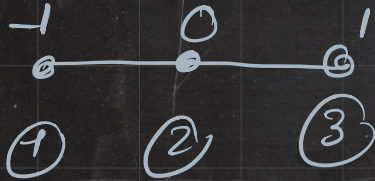
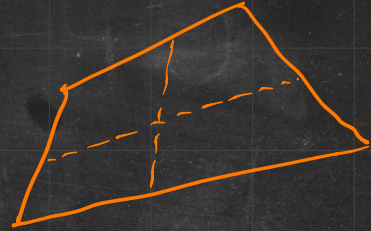


$$O(f(x)) \leq O(g(\xi))$$

CONSTANT $J \Rightarrow$ IDENTICAL ORDERS FOR $f(x)$ & $g(\xi)$



x

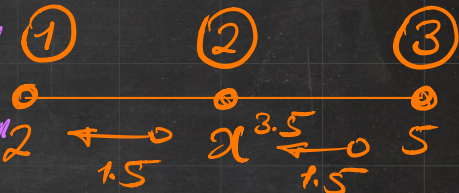


ξ IN 2D

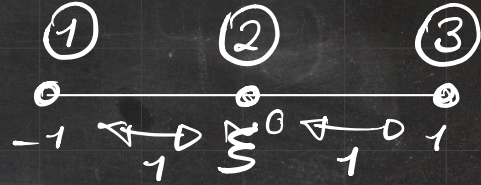
ξ

NATURAL SPACE IS "GENERALLY" UNIFORMLY DISCRETIZED

↳ Physical space is the result of geometry
 ↳ THIS IS NOT THE CASE FOR discretization



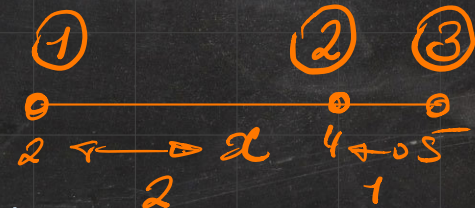
PHYSICAL SPACE NECESSARILY



J
 Δ
 UNIFORM
 (CONST.)

↳ WHY IS IT NOT EASY?

J UNIFORM IS DESIRABLE!



J
 Δ
 $J(\xi)$

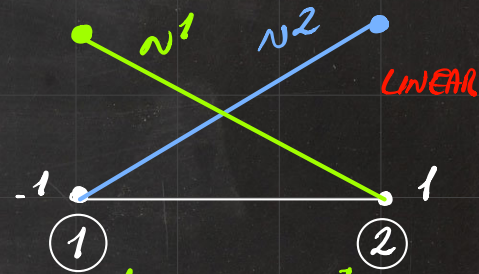
J : CONSTANT $\rightarrow$ ORDER OF FLUX
 AND $g(\xi)$ ARE THE SAME!

EXAMPLE: $\int_2^5 2x^2 + x - 1 dx = ?$

* ANALYTICAL SOLUTION

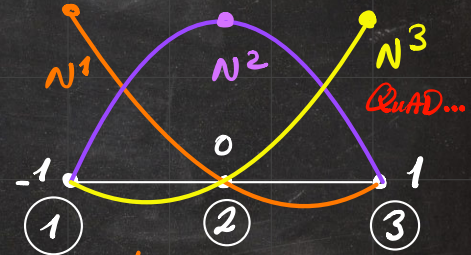
$$\int_2^5 2x^2 + \frac{1}{2}x^2 - x \Big|_2^5 = 000$$

$$= 85.5$$



$$N^1 = -\frac{1}{2}[\xi - 1]$$

$$N^2 = \frac{1}{2}[\xi + 1]$$



$$N^1 = \frac{1}{2}\xi[\xi - 1]$$

$$N^2 = [1 - \xi][1 + \xi]$$

$$N^3 = \frac{1}{2}\xi[\xi + 1]$$

EXAMPLE: $\int_2^5 2x^2 + x - 1 dx = ?$

* ANALYTICAL SOLUTION $\rightarrow 85.5$

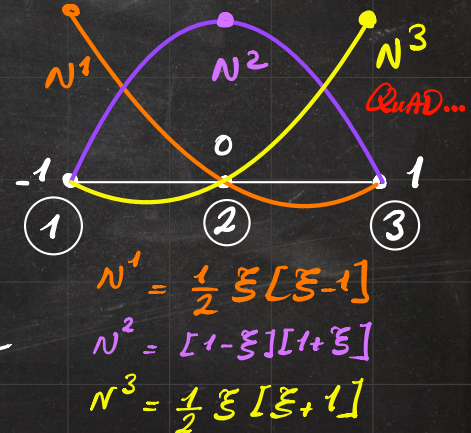
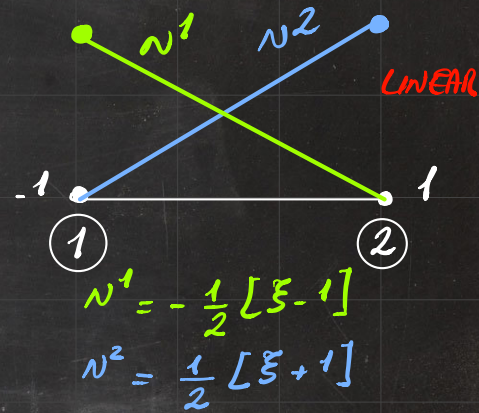
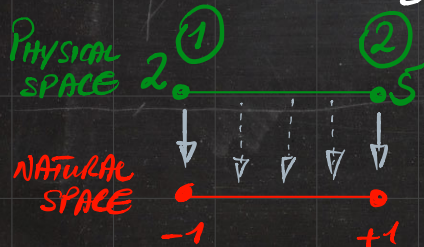
* APPROXIMATE SOLUTION $\rightarrow$ LINEAR

$$x = x(\xi) = N^1 x^1 + N^2 x^2$$

$$= -\frac{1}{2}[\xi - 1] \times 2 + \frac{1}{2}[\xi + 1] \times 5$$

$$= 1.5\xi + 3.5 \rightarrow x(\xi)$$

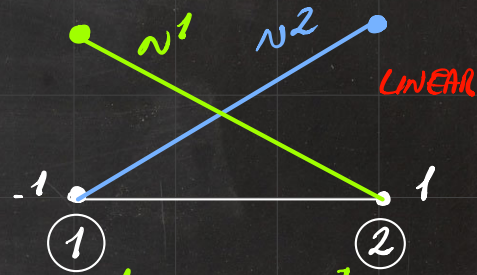
$$dx = 1.5 d\xi \rightarrow J = 1.5$$



EXAMPLE: $\int_2^5 2x^2 + x - 1 dx = ?$

* ANALYTICAL SOLUTION $\rightarrow 85.5$

* APPROXIMATE SOLUTION $\rightarrow$ LINEAR



$$N^1 = -\frac{1}{2}[\xi - 1]$$

$$N^2 = \frac{1}{2}[\xi + 1]$$

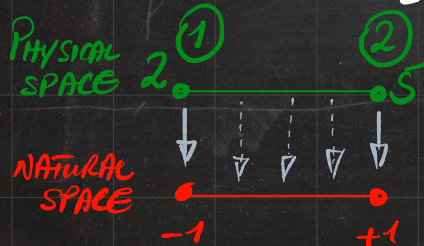
$$x = x(\xi) = N^1 x^1 + N^2 x^2$$

$$= -\frac{1}{2}[\xi - 1] \times 2 + \frac{1}{2}[\xi + 1] \times 5$$

$$= 1.5\xi + 3.5 \rightarrow x(\xi)$$

$$dx = 1.5 d\xi \rightarrow J = 1.5$$

$$x = 1.5\xi + 3.5$$



$$\begin{aligned} & \int_2^5 2x^2 + x - 1 dx \quad \text{Quad.} \\ & \quad \quad \quad \downarrow \text{Approx.} \\ & = \int_{-1}^1 g(\xi) d\xi \quad \text{GPE} \\ & \quad \quad \quad \downarrow \\ & \quad \quad \quad \xi = 2 \\ & = g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right) \\ & = 23.264 + 62.236 \\ & = 85.500 \checkmark \end{aligned}$$

EXAMPLE: $\int_2^5 2x^2 + x - 1 dx = ?$ $\rightarrow x = 1.5\xi + 3.5$

* APPROXIMATE SOLUTION $\rightarrow$ QUADRATIC UNIFORM

$$x = x(\xi) = N^1 x^1 + N^2 x^2 + N^3 x^3$$

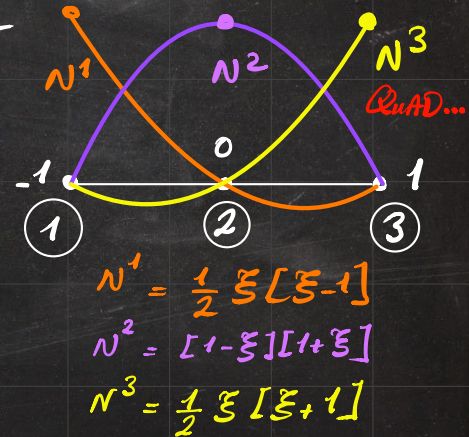
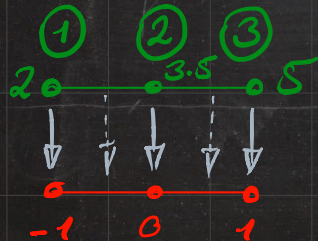
$dx = 1.5 d\xi$

$\int_{-1}^1 g(\xi) d\xi \rightarrow$ QUAD. POLY. $\downarrow$ GPE=2

$$= \frac{1}{2} \xi [\xi - 1] x^2 + [1 - \xi][1 + \xi] x^3 + \frac{1}{2} \xi [\xi + 1] x^5$$

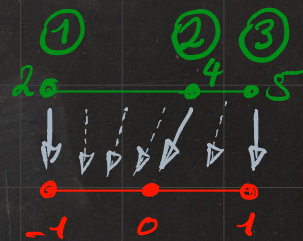
$$= \dots = 1.5 \xi + 3.5$$

$$J = 1.5 \rightarrow dx = J d\xi$$



EXAMPLE: $\int_2^5 \underbrace{2x^2 + x - 1}_{\text{Quadratic}} dx = ? \quad \leftarrow x = -0.5\xi^2 + 1.5\xi + 4$

* APPROXIMATE SOLUTION $\rightarrow$ QUADRATIC Non-uniform



$$x = N^1 x^1 + N^2 x^2 + N^3 x^3$$

$$\int_{-1}^1 g(\xi) d\xi$$

$\leftarrow dx = [-\xi + 1.5] d\xi$
 $\rightarrow$ 3GP: 85.5

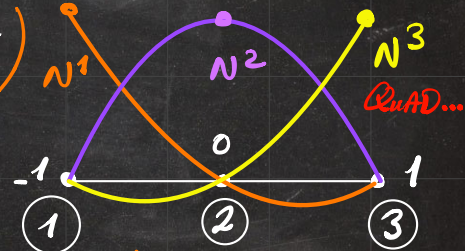
$$= \frac{1}{2} \xi [\xi - 1] x^1 + [1 - \xi] [1 + \xi] x^2 + \frac{1}{2} \xi [\xi + 1] x^3$$

$$= 0 = -0.5 \xi^2 + 1.5 \xi + 4$$

$\Downarrow$

$$dx = [-\xi + 1.5] d\xi \Rightarrow J = -\xi + 1.5$$

2GP: 84.834



$$N^1 = \frac{1}{2} \xi [\xi - 1]$$

$$N^2 = [1 - \xi] [1 + \xi]$$

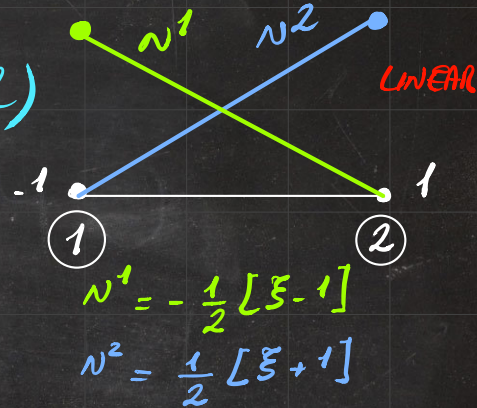
$$N^3 = \frac{1}{2} \xi [\xi + 1]$$

EXAMPLE:

ANALYTICAL $\rightarrow \left[\frac{x^4}{4} + \frac{x^3}{3} \right]_2^5 = 191.25$

$$\int_2^5 x^3 + x^2 dx$$

APPROXIMATE (LINEAR)



$$x(\xi) = N^1 x^1 + N^2 x^2$$

$$= -\frac{1}{2}[\xi - 1] \times 2 + \frac{1}{2}[\xi + 1] \times 5$$

$$\Rightarrow x = x(\xi) = 1.5\xi + 8.5 \Rightarrow dx = 1.5 d\xi \Rightarrow J = 1.5$$

EXAMPLE:

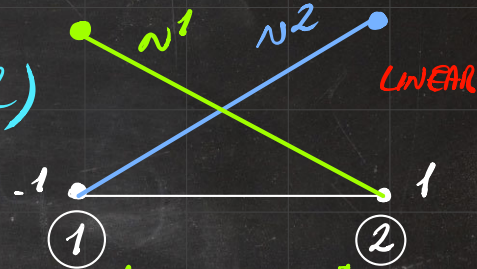
ANALYTICAL $\rightarrow \left[\frac{x^4}{4} + \frac{x^3}{3} \right]_2^5 = 191.25$

$\int_2^5 x^3 + x^2 dx$ APPROXIMATE (LINEAR)

$x = x(\xi) = 1.5\xi + 3.5 \quad J = 1.5$

$\int_{-1}^1 \left[[1.5\xi + 3.5]^3 + [1.5\xi + 3.5]^2 \right] 1.5 d\xi$

GPE = 2 $\rightarrow$ ^{Cubic} Poly $\rightarrow g(\xi)$



$N^1 = -\frac{1}{2}[\xi - 1]$

$N^2 = \frac{1}{2}[\xi + 1]$

$= g(-\frac{1}{\sqrt{3}}) + g(\frac{1}{\sqrt{3}})$

$= 37.818 + 153.43$
 $= 191.25 \checkmark$